Differentiation Examples and Rules

1. The power functions z^n, $n = 0, \pm 1, \pm 2, \ldots$, are analytic and

$$\frac{dz^n}{dz} = nz^{n-1}.$$

2. The conjugate function, $g(z) = \bar{z}$, is *not* differentiable for any z.
Note that

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

does not exist.

3. **Cauchy – Riemann Equations:** If $G(x, y) = u(x, y) + iv(x, y)$ is a complex valued function of two variable with continuous partial derivatives, then $G(z) = G(x + iy)$ is analytic if and only if

$$\frac{\partial G}{\partial x} = -i \frac{\partial G}{\partial y} = G'(z).$$

In terms of real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

4. Verify that the following functions satisfy the Cauchy – Riemann Equations:

- $$\mathbf{exp}(x + iy) \stackrel{\text{def}}{\equiv} e^x \left(\cos(y) + i \sin(y)\right).$$

- $$\mathbf{Ln}(x + iy) \stackrel{\text{def}}{\equiv} \ln|x + iy| + i \arctan\left(\frac{y}{x}\right), x > 0.$$

5. Verify that

- $$\frac{d\mathbf{exp}(z)}{dz} = \mathbf{exp}(z).$$

- $$\frac{d\,\mathbf{Ln}(z)}{dz} = \frac{1}{z}, \Re z > 0.$$

6. **The Chain Rule.** Let $g(z)$ be analytic at z, and let $f(w)$ be analytic at $w = g(z)$. Then $h(z) = f(g(z))$ is analytic at z and

$$\frac{d}{dz} f(g(z)) = f'\left(g(z)\right) \cdot g'(z).$$

Proof: We are assuming that $g(z + \Delta z) = g(z) + g'(z) \cdot \Delta z + o\left(\Delta z\right)$ and $f(g(z) + \Delta g(z)) = f(g(z)) + f'(g(z)) \cdot \Delta g(z) + o\left(\Delta g(z)\right)$. Since $\Delta g(z) = g'(z)\Delta z + o\left(\Delta z\right)$, and $o\left(\Delta g(z)\right) = o\left(O\left(\Delta z\right)\right)$, we have $f(g(z + \Delta z)) = f(g(z)) + f'(g(z)) \cdot g'(z) \cdot \Delta z + o\left(\Delta z\right)$.