EXPERIMENTAL

Cauchy's Integral Theorem

The fundamental result - **Cauchy's Integral Theorem** - says roughly:
If C is a simple closed path and $w = f(z)$ is analytic inside and on C, then

$$\oint_C f(z)\,dz = 0.$$

There are two common approaches to this result. The first approach uses the Cauchy–Riemann equations and Green's Theorem. The second approach uses less assumptions about the regularity of the derivative f' and builds up the proof by first considering C to be a simple closed triangle and then approximating the general simple closed path by a simple closed polygonal path.

Cauchy's Integral Theorem using Green's Formula

Theorem. *Let C be a simple closed path enclosing a region D. Suppose that on $D \cup C$, $w = f(z)$ is analytic and that f' is continuous. Then*

$$\oint_C f(z)\,dz = 0.$$

Proof: By the Cauchy–Riemann Equations,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Then by Green's Formula

$$\begin{aligned} 0 = \int\int_D \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy &= \oint_C f\,dy - i\oint_C f\,dx \\ &= -i\oint_C f\ (dx + idy) \\ &= -i\oint_C f(z)\,dz \\ &= 0. \end{aligned}$$

Consequences of Cauchy's Integral Theorem

1. **In Simply Connected Regions Integrals of Analytic functions are Independent of the Path**

A region D is *simply connected* if for every simple closed path C in D, all of the points inside C are also in D. The most important examples of simply connected regions are

- Circles: $\{z|\,|z - z| < R\}$
- Half Planes: $\{z|\Re z > 0\}$
- The whole complex plane $\mathbf{C}$
- Convex regions

Let $w = f(z)$ be analytic in a simply connected region D. Let Z_1 and Z_2 be two points in D, and take two paths C_1 and C_2 in D which go from Z_1 (initial point) to Z_2 (terminal point). Then $C_1 - C_2$ can be broken into simple closed paths so that

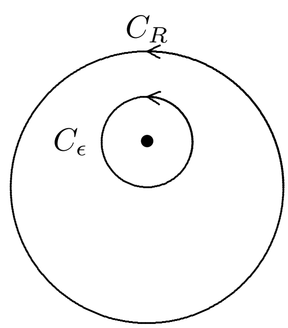
$$0 = \int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz,$$
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Thus we define

$$\int_{Z_1}^{Z_2} f(z)\,dz = \int_C f(z)\,dz,$$

where C is *any* path in D from Z_1 to Z_2.

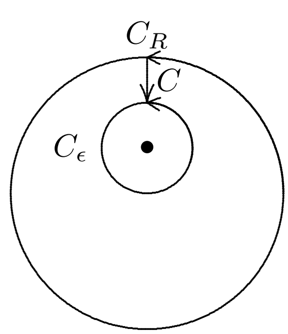
2. **Two Circles Theorem**
Let C_ϵ be a circle inside a circle C_R.



Suppose that $f(z)$ is analytic on the two circles and the region in between the two circles. Then

$$\oint_{C_\epsilon} f(z)\,dz = \oint_{C_R} f(z)\,dz.$$

. The proof uses a cut C from the outer circle to the inner circle.



Then

$$\oint_{C_R} f(z)\,dz - \oint_{C_\epsilon} f(z)\,dz = \oint_{C_R + C - C_\epsilon - C} f(z)\,dz$$
$$= 0.$$

3. **Fundamental Theorem of Calculus Version II**
   Let $w = f(z)$ be analytic in a simply connected region D. For $z \in D$, define

$$F(z) = \int_{Z_0}^{z} f(\zeta)\,d\zeta.$$

Then $F(z)$ in analytic in D and $F'(z) = f(z)$.