Cauchy’s Integral Theorem
The fundamental result - Cauchy’s Integral Theorem - says roughly:
If $C$ is a simple closed path and $w = f(z)$ is analytic inside and on $C$, then
\[ \oint_C f(z) \, dz = 0. \]

There are two common approaches to this result. The first approach uses the Cauchy–Riemann equations and Green’s Theorem. The second approach uses less assumptions about the regularity of the derivative $f'$ and builds up the proof by first considering $C$ to be a simple closed triangle and then approximating the general simple closed path by a simple closed polygonal path.

Cauchy’s Integral Theorem using Green’s Formula

**Theorem.** Let $C$ be a simple closed path enclosing a region $D$. Suppose that on $D \cup C$, $w = f(z)$ is analytic and that $f'$ is continuous. Then
\[ \oint_C f(z) \, dz = 0. \]

**Proof:** By the Cauchy–Riemann Equations,
\[ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = 0. \]

Then by Green’s Formula
\[
0 = \iint_D \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \, dx \, dy = \oint_C f \, dy - i \oint_C f \, dx \\
= -i \oint_C f \, (dx + idy) \\
= -i \oint_C f(z) \, dz \\
= 0.
\]

Consequences of Cauchy’s Integral Theorem

1. **In Simply Connected Regions Integrals of Analytic functions are Independent of the Path**
   A region $D$ is *simply connected* if for every simple closed path $C$ in $D$, all of the points inside $C$ are also in $D$. The most important examples of simply connected regions are:
   - Circles: $\{ z | |z - z'| < R \}$
   - Half Planes: $\{ z | \Re z > 0 \}$
   - The whole complex plane $\mathbb{C}$
   - Convex regions

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Let \( w = f(z) \) be analytic in a simply connected region \( D \). Let \( Z_1 \) and \( Z_2 \) be two points in \( D \), and take two paths \( C_1 \) and \( C_2 \) in \( D \) which go from \( Z_1 \) (initial point) to \( Z_2 \) (terminal point). Then \( C_1 - C_2 \) can be broken into simple closed paths so that

\[
0 = \int_{C_1 - C_2} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz, \\
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]

Thus we define

\[
\int_{Z_1}^{Z_2} f(z) \, dz = \int_C f(z) \, dz,
\]

where \( C \) is any path in \( D \) from \( Z_1 \) to \( Z_2 \).

2. Two Circles Theorem

Let \( C_\epsilon \) be a circle inside a circle \( C_R \).

Suppose that \( f(z) \) is analytic on the two circles and the region in between the two circles. Then

\[
\oint_{C_\epsilon} f(z) \, dz = \oint_{C_R} f(z) \, dz.
\]

The proof uses a cut \( C \) from the outer circle to the inner circle.
Then
\[ \oint_{C_R} f(z) \, dz - \oint_{C_\epsilon} f(z) \, dz = \oint_{C_R + C_\epsilon - C} f(z) \, dz = 0. \]

3. **Fundamental Theorem of Calculus Version II**

Let \( w = f(z) \) be analytic in a simply connected region \( D \). For \( z \in D \), define

\[ F(z) = \int_{z_0}^{z} f(\zeta) \, d\zeta. \]

Then \( F(z) \) is analytic in \( D \) and \( F'(z) = f(z) \).