Cauchy's Integral Theorem – Proof II

The fundamental result - **Cauchy's Integral Theorem** - says roughly:
If C is a simple closed path and $w = f(z)$ is analytic inside and on C, then

$$\oint_C f(z)\,dz = 0.$$

There are two common approaches to this result. The first approaches the Cauchy–Riemann equations and Green's Theorem. The second approach uses less assumptions about the regularity of the derivative f' and builds up the proof by first considering C to be a simple closed triangle and then approximating the general simple closed path by a simple closed polygonal path.

Cauchy's Integral Theorem for a Triangle Using Bolzano–Weierstraß

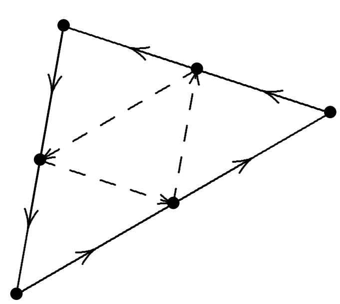
Theorem. *Let T be a simple closed triangle enclosing a region D. Suppose that on $D \cup T$, $w = f(z)$ is analytic. (No assumption is made on the continuity of $f'(z)$.) Then*

$$\oint_T f(z)\,dz = 0.$$

Proof: The proof is by contradiction. Suppose there is a triangle T_0 for which

$$\left|\oint_{T_0} f(z)\,dz\right| = \delta > 0.$$

Let the perimeter of T_0 be p_0. Using the midpoints of eachside of T_0, divide T_0 into four congruent triangles, each of perimeter $p_1 = \frac{p_0}{2^1}$.



On at least one of the four triangles, call it T_1,

$$\left|\oint_{T_1} f(z)\,dz\right| \geq \frac{\delta}{4^1}.$$

Next, using the midpoints of eachside of T_1, divide T_1 into four congruent triangles, each of perimeter $p_2 = \frac{p_0}{2^2}$. On at least one of the four triangles, call it T_2,

$$\left|\oint_{T_2} f(z)\,dz\right| \geq \frac{\delta}{4^2}.$$

Continuing in this manner, we construct a shrinking sequence of similar triangles, T_n, of perimeter $p_n = \frac{p_0}{2^n}$. with

$$\left| \oint_{T_n} f(z)\, dz \right| \geq \frac{\delta}{4^n}.$$

By the Bolzano–Weierstraß Property, there is a number z_0 within all the triangles. Since $f(z)$ is analytic at $z = z_0$, near z_0,

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + o\,(z - z_0)\,.$$

Thus

$$\begin{aligned}
\oint_{T_n} f(z)\, dz &= \oint_{T_n} f(z_0) + f'(z_0) \cdot (z - z_0) + o\,(z - z_0)\ dz \\
&= 0 + 0 + \oint_{T_n} o\,(z - z_0)\ dz \\
&= o\left(\frac{1}{2^n}\right) \frac{p_0}{2^n} \\
&= o\left(\frac{1}{4^n}\right).
\end{aligned}$$

But $\frac{\delta}{4^n}$ is not $o\left(\frac{1}{4^n}\right)$.

If P is a simple closed polygon, it follows that if $f(z)$ is analytic on and inside P,

$$\oint_P f(z)\, dz = 0.$$

The case of C being a simple closed oriented piecewise C^1 path follows by approximating C by simple closed polygons.