Limits and Order

For functions of a real variable, the derivative is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

which means that the difference

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x)$$

is *small* if Δx is small and not 0 (for which the quotient is not obviously defined).

Multiplying the remainder by Δx, we obtain that

$$f(x+\Delta x) - f(x) - f'(x)\Delta x = \mathit{small} \cdot \Delta x,$$

with the right hand side, (**RHS**), of the equation is "*much smaller than* Δx", as $\Delta x \to 0$, in the precise sense

$$\lim_{\Delta x \to 0} \frac{\textbf{RHS}}{|\Delta x|} = 0.$$

Another formal advantage is that the equation is also defined and true for $\Delta x = 0$.

Definition. As $\Delta x \to 0$, an expression $\phi(\Delta x)$ is *little o of* Δx, written $o(\Delta x)$, if

$$\lim_{\Delta x \to 0} \frac{\phi(\Delta x)}{|\Delta x|} = 0.$$

If we are not worried about the particular details of $\phi(x)$, we write $\phi(x) = o(\Delta x)$. With this convention, the definition of differentiability and the derivative takes the convenient form

$$f(x+\Delta x) = f(x) + f'(x) \cdot \Delta x + o(\Delta x).$$

In a similar way, if $\lim_{\Delta x \to 0} \psi(\Delta x) = 0$, we write $\psi(\Delta x) = o(1)$ with the precise meaning that

$$\lim_{\Delta x \to 0} \frac{\psi(\Delta x)}{1} = 0.$$

Definition. *Let $q(\Delta x)$ be nonzero for Δx near 0. Then a function $\phi(\Delta x)$ is little o of $q(\Delta x)$, written $\phi(\Delta x) = o(q(\Delta x))$, if*

$$\lim_{\Delta x \to 0} \frac{\phi(\Delta x)}{|q(\Delta x)|} = 0.$$

Then a function $\phi(x)$ *is big O of* $q(\Delta x)$*, written* $\phi(\Delta x) = O\left(q(\Delta x)\right)$*, if*

$$\frac{\phi(\Delta x)}{|q(\Delta x)|}$$

is bounded as $\Delta x \to 0$*.*

With this convention, continuity of a function $f(x)$ can be expressed by

$$f(x + \Delta x) = f(x) + o\left(1\right),$$

and local boundedness of a function can be expressed as $f(x + \Delta x) = O\left(1\right)$.

There is a formal calculus for handling sums and products for functions which are *little o* or *big O* of one (or several) q. Verify that $O\left(1\right) \cdot o\left(\Delta x\right) = o\left(\Delta x\right)$; i.e., the product of a bounded function and a function which is $o\left(\Delta x\right)$ is $o\left(\Delta x\right)$. Similarly $o\left(\Delta x\right) \pm o\left(\Delta x\right) = o\left(\Delta x\right)$.

The concepts *little o* and *big O* are also useful as the argument $x \to \infty$. For example we write $x^2 = o\left(e^x\right)$ as $x \to \infty$ with the precise meaning

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0.$$