**Paths and Integrals**

A $C^1$ *path* $C$ is a complex valued function

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b,$$

where $z(t)$ is continuously differentiable. The path $C$ is represented by its image with an arrow drawn in the direction of increasing $t$.

We shall assume the path is *simple* - it does not intersect itself except possibly at its endpoints. The path is *closed* if it is simple and the endpoints are the same: $z(a) = z(b)$.

If $C$ is a path, the path $-C$ is the path with the same image but traced in the opposite direction. If $C$ is parameterized by $z_C(t), 0 \leq t \leq 1$, then $-C$ may be parameterized by

$$-C : z = z_{-C}(t) = z_C(1-t), 0 \leq t \leq 1.$$}

We shall deal with paths which are continuous and piecewise $C^1$. Such paths can written as a formal sum $C_1 + C_2 + \ldots + C_N$, where the terminal point of $C_j$ is the initial point of $C_{j+1}, j = 1, \ldots, N - 1$.  

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1 More generally, we can consider a *chain*: $C_1 + C_2 + \ldots + C_N$, a formal sum even when the components do not connect. The concept of *chain* is used in differential geometry.
For our purposes $C$ will consist of a [small] number of arcs and line segments.

**Integrals on Paths**

Let $C$ be a continuous and [piecewise] $C^1$ path and let $f(z)$ be a continuous function defined on $C$. Let $C$ be parameterized by

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b.$$ 

Then the integral of $f(z) \, dz$ on $C$ is defined as:

- **The Quick Definition**

  $$\int_C f(z) \, dz = \int_a^b f(z(t)) \frac{dz}{dt} \, dt,$$

  where $C$ is parameterized by

  $$C : z = z(t) = x(t) + iy(t), a \leq t \leq b.$$

- **The Riemann Sum Definition:**

  Let $\Pi : a = t_0 < t_1 < \ldots < t_M = b$, be a partition of $[a,b]$, and $z'_j = z(t'_j)$ be a typical point in the image of $[t_j, t_{j+1}]$; define the *Riemann sum*

  $$R(f, \Pi, z'_j) = \sum_{j=0}^{M-1} f(z'_j) (z_{j+1} - z_j)$$

  $$\approx \sum_{j=0}^{M-1} f(z'_j) \cdot z'(t'_j) \cdot (t_{j+1} - t_j)$$

  $$= \sum_{z \text{along } C} f(z) \Delta z.$$

  Then

  $$\int_C f(z) \, dz = \lim_{\max(\Delta z) \to 0} R(f, \Pi, z'_j)$$

  $$= \lim_{\max(\Delta z) \to 0} \sum_{z \text{along } C} f(z) \Delta z.$$
The “quick” definition gives an effective way to compute $\int_C f(z) \, dz$. The “Riemann sum” definition emphasizes that $\int_C f(z) \, dz$ is independent of the parameterization of $C$. Either definition gives that

$$\int_C f(z) \, dz = \int_C f(z) \, dz \pm \int_C g(z) \, dz,$$

$$\int_{-C} f(z) \, dz = -\int_C f(z) \, dz,$$

$$\int_{C_1+C_2} f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz.$$

The last relation says that for fixed $f(z)$, $\int_C f(z) \, dz$ is additive as a map on sums of paths or chains.\(^2\)

**A Version of the Fundamental Theorem of Calculus**

**Theorem (FTC Version I).** Let $C$ be a continuous piecewise $C^1$ path and let $F(z)$ be analytic at every point on $C$. Then

$$\int_C F'(z) \, dz = F(z(b)) - F(z(a)).$$

where

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b.$$

is a parameterization of $C$.

**Proof:** By the chain rule for differentiation

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \frac{dz}{dt},$$

so by FTC (Version I) for functions of a real variable

$$\int_C F'(z) \, dz = \int_a^b \frac{d}{dt} F(z(t)) \, dt$$

$$= F(z(b)) - F(z(a)).$$

**The Most Important Path Integral**

If the curve $C$ is simple and closed and traversed in the *counterclockwise* direction, we often write

$$\int_C f(z) \, dz = \oint_C f(z) \, dz$$

The most important integral is the integral of $f(z) = \frac{1}{z}$ around a circle containing the origin.

\(^2\) Compare to the calculus result

$$\int_a^b f(t) \, dt + \int_b^c f(t) \, dt = \int_a^c f(t) \, dt.$$
**Theorem.** Let $C_R$ be the circle of radius $R$, centered at 0, and traversed in the counterclockwise direction. Then

$$\oint_{C_R} \frac{1}{z} \, dz = 2\pi i.$$  

**Proof:** $C_R$ can be parameterized by the angle $t$, $0 \leq t \leq 2\pi$:

$$C_R : z(t) = Re^{it} = R(\cos(t) + i \sin(t)),$$

$$dz = iRe^{it} \, dt.$$

Then

$$\oint_{C_R} \frac{1}{z} \, dz = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}} \, dt = \int_0^{2\pi} i \, dt = 2\pi i.$$

**Exercise**

Use the above Fundamental Theorem Calculus or the parametric representation to show that for $n$ an integer, $n \neq -1$,

$$\oint_{C_R} z^n \, dz = 0.$$