Complex Power Series

Convergence and Absolute Convergence

- $\sum_{*}^{\infty} C_n$ CONVerges iff

$$\lim_{N\to\infty} \sum_{*}^{N} C_n$$

  exists (finite).

- If $C_n = x_n + iy_n$, then $\sum_{*}^{\infty} C_n$ CONVerges iff $\sum_{*}^{\infty} x_n$ CONVerges and $\sum_{*}^{\infty} y_n$ CONVer

- $\sum_{*}^{\infty} C_n$ CONVerges ABSsolutely iff $\sum_{*}^{\infty} |C_n|$ converges or

$$\lim_{N\to\infty} \sum_{*}^{N} |C_n|$$

  exists (finite).

- If $C_n = x_n + iy_n$, then $\sum_{*}^{\infty} C_n$ CONVerges ABSolutely iff $\sum_{*}^{\infty} x_n$ and $\sum_{*}^{\infty} y_n$ CONVe ABSolutely.

- $\sum_{*}^{\infty} |C_n|$ converges iff the sequence $\sum_{*}^{N} |C_n|$ is bounded.

Comparison Test for Absolute Convergence

- If

$$|C_n| \leq B_n,$$

  then

$$0 \leq \sum_{*}^{\infty} |C_n| \leq \sum_{*}^{\infty} B_n.$$

  So that if the series $\sum_{*}^{\infty} B_n$ CONVerges, the $\sum_{*}^{\infty} A_n$ CONVerges ABSolutely. Moreo

there is the obvious error estimate

$$\left| \sum_{N+1}^{\infty} C_n \right| \leq \sum_{N+1}^{\infty} |C_n|$$

The Geometric Series

We consider the geometric series

$$\sum_{0}^{\infty} z^n.$$

A bare hands calculation shows that

$$\begin{aligned}(1-z)\sum_{0}^{N} z^n &= (1-z)\left(1+z+\ldots+z^N\right)\\ &= 1-z^{N+1}.\end{aligned}$$

so that

$$\sum_{0}^{\infty} z^n \quad \begin{cases} \text{CONVerges ABSolutely to } \frac{1}{1-z} \text{ for } |z|<1, \\ \text{DIVerges for } |z| \geq 1. \end{cases}$$

For $|z| < r < 1$, there is the error estimate

$$\begin{aligned}\left|\sum_{N+1}^{\infty} z^n\right| \leq \sum_{N+1}^{\infty} |z|^n &= \frac{|z|^{N+1}}{1-|z|}\\ &\leq \left|\frac{z}{r}\right|^{N+1} \frac{r^{N+1}}{1-r}\\ &\leq \frac{r^{N+1}}{1-r}.\end{aligned}$$

Ratio Test for ABSolute CONVergence

For the series $\sum_{*}^{\infty} C_N$, suppose that

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = L.$$

- If $0 \leq L < 1$, the series $\sum_{*}^{\infty} C_N$ CONVerges ABSolutely.

Why? Choose an r, $L < r < 1$. For n sufficiently large, $|C_{n+1}| < r\,|C_n|$ and for N la
enough,

$$\begin{aligned}\sum_{N}^{\infty} |C_n| &\leq \sum_{N}^{\infty} |C_N|\, r^{n-N}\\ &= |C_N|\,\frac{1}{1-r}.\end{aligned}$$

- If $L = 1$, we are not sure – additional information is needed to decide DIVergence
CONVergence and/or ABS0lute CONVergence.

Power Series, Radius of Convergence, and Circle/Disk of Convergence

- If the power series $\sum_{n=0}^{\infty} a_n z^n$, converges for a nonzero $z = z_0$, then for all z, $|z| < |z_0|$,
power series CONVerges ABSolutely.

Why? We have that $\lim_{n\to\infty} |a_n z_0^n| = 0$. Then we can compare

$$\begin{aligned}\sum_{N+1}^{\infty} |a_n z^n| &= \sum_{N+1}^{\infty} |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \\ &\leq \left(\max_{n\geq N} |a_n z_0^n|\right) \sum_{N+1}^{\infty} \left|\frac{z}{z_0}\right|^n \\ &= o(1) \frac{\theta^{N+1}}{1-\theta},\end{aligned}$$

where $o(1)$ means $\lim_{N\to\infty} o(1) = 0$, and $\theta = \left|\frac{z}{z_0}\right|$.

Thus the convergence of the series at a nonzero z_0 forces the absolute convergence of
series in the entire open disk centered at 0 with radius $|z_0|$.

- For a power series $\sum_{n=0}^{\infty} a_n z^n$, there is a number R, $0 \leq R \leq \infty$ for which

$$\sum_{n=0}^{\infty} a_n z^n \begin{cases} \text{CONVerges ABSolutely for } |z| < R, \\ \text{DIVerges for } |z| > R. \end{cases}$$

The number R is called the *radius of convergence* of the power series. R can often
determined by the Ratio Test.

- If $f(z)$ is represented by a convergent power series for $|z| < R$, then $f(z)$ is an analy
function in the region $|z| < R$ and its derivative is represented by the convergent se
$\sum_{n=1}^{\infty} n a_n z^{n-1}$, $|z| < R$.

Thus the power series for f' has radius of convergence at least R, and the form
differentiated series converges to the analytic function $f'(z)$. Within the open disk
convergence, it follows that function represented by a power series has derivatives of
orders which are represented by the series differentiated term by term.

then

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, |x| < R,$$

and[1]

$$\int_0^z f(\zeta)\, d\zeta = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n, |z| < R$$