**Residues at Isolated Singularities**

If $z_0$ is a complex number we shall use the notation $C_r(z_0)$ for the closed circular path

$$C_r(z_0) = \{z \mid |z - z_0| = r \}$$

traversed in the counterclockwise direction.

**Residue Theorem – One Singularity Version.** Let $f(z)$ be analytic in the region

$$\{ z \mid 0 < |z - z_0| < R \}.$$ Then for $z$ inside $C_r(z_0)$, $z \neq z_0$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta) (\zeta - z_0)^{-n-1} \, d\zeta.$$ Here $r$ is any number such that $0 < r < R$.

In particular

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta) \, d\zeta,$$

is the residue of $f(z)$ at $z = z_0$. so that

$$\oint_{C_r(z_0)} f(\zeta) \, d\zeta = 2\pi i \text{Res} \, f(z) \bigg|_{z = z_0}.$$ **The Residue Theorem**

Let $C$ be a simple closed path. Suppose that $f(z)$ is analytic on and inside $C$, except for a finite number of isolated singularities, $z_1, z_2, \ldots, z_K$ inside $C$.

Then, by using cuts from $C$ to small circles of small radius $r$ around each $z_k$,
\[
\oint_C f(\zeta) \, d\zeta = \sum_{k=1}^{K} \oint_{C_r(z_k)} f(\zeta) \, d\zeta \\
= \sum_{k=1}^{K} 2\pi i \operatorname{Res} f(z) \big|_{z=z_k} \\
= 2\pi i \sum_{k=1}^{K} \operatorname{Res} f(z) \big|_{z=z_k}.
\]

**The Residue Theorem.** Let \( C \) be a simple closed path. Suppose that \( f(z) \) is analytic on and inside \( C \), except for a finite number of isolated singularities, \( z_1, z_2, \ldots, z_K \) inside \( C \). Then

\[
\oint_C f(\zeta) \, d\zeta = 2\pi i \sum_{k=1}^{K} \operatorname{Res} f(z) \big|_{z=z_k}.
\]

The Residue Theorem reduces the problem of evaluating a *contour integral* – an integral on a simple closed path – to the algebraic problem of determining the poles and residues\(^1\) of a function.

**Exercises**

Note the following special cases:

1. Let \( C \) be a simple closed path. Suppose that \( f(z) \) is analytic on and inside \( C \). Use the Residue Theorem to show that

\[
\oint_C f(\zeta) \, d\zeta = 0.
\]

We knew this result already!

2. Let \( C \) be a simple closed path. Let \( z \) be a point inside \( C \). Find

\[
\oint_C \frac{1}{\zeta - z} \, d\zeta.
\]

3. Let \( C \) be a simple closed path. Let \( z \) be a point outside \( C \). Find

\[
\oint_C \frac{1}{\zeta - z} \, d\zeta.
\]

4. Let \( C \) be a simple closed path. Let \( a \) and \( b \) be points inside \( C \), \( a \neq b \). Find

\[
\oint_C \frac{1}{(\zeta - a)(\zeta - b)} \, d\zeta.
\]

**Finding the Residue**

\(^1\) Since functions behave so badly at an essential singularity, there is little hope of finding the residue at an essential singularity.
If $f(z)$ has a pole of order $M$ at $z = z_0$, then

$$f(z) = \frac{a_{-M}}{(z - z_0)^M} + \ldots + \frac{a_{-1}}{z - z_0} + \ldots,$$

$$(z - z_0)^M f(z) = a_{-M} + a_{-M+1} (z - z_0) + \ldots + a_{-1} (z - z_0)^{M-1} + \ldots.$$ 

Since $(z - z_0)^M f(z)$ is analytic at $z = z_0$,

$$a_{-1} = \frac{1}{(M - 1)!} \frac{d^{M-1}}{dz^{M-1}} (z - z_0)^M f(z) \bigg|_{z=z_0}.$$ 

**Partial Fractions**

To find the partial fraction expansion of

$$f(z) = \frac{q_{M+N-1}(z)}{(z - z_0)^M p_N(z)},$$

degree $q_{M+N-1} = M + N - 1$,

degree $p_N = N$,

$q_{M+N-1}(z_0) \neq 0$,

$p_N(z_0) \neq 0$,

let

$$h(z) = \frac{q_{M+N-1}(z)}{p_N(z)}.$$ 

Near $z = z_0$,

$$f(z) = \frac{h(z_0)}{0!} \frac{1}{(z - z_0)^M} + \frac{h'(z_0)}{1!} \frac{1}{(z - z_0)^{M-1}}$$

$$+ \ldots + \frac{h^{(M-1)}(z_0)}{(M-1)!} \frac{1}{z - z_0} + \text{analytic}.$$ 

Similar expansions may be found near the roots of $p_N(z) = 0$. 

residue.tex