Sums and Unconditional Convergence

For manipulations with power series we are interested in infinite sums such as

$$\sum_{j,k=0}^{\infty} a_{j,k}.$$  

We need to justify changing the order of summation and conditions which assure that

$$\sum_{j,k=0}^{\infty} a_{j,k} = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{j,k} \right) = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{j,k} \right).$$

The condition which justifies the manipulations is absolute or unconditional convergence of the series.

**Theorem.** Suppose that all the finite [partial] sums

$$\sum_{j,k \in \text{finite set}} |a_{j,k}|$$

form a bounded set. Then there is a unique finite (complex) number

$$S = \sum_{j,k=0}^{\infty} a_{j,k}$$

which may be calculated as

$$\sum_{j,k=0}^{\infty} a_{j,k} = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{j,k} \right) = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{j,k} \right),$$

or with any rearrangement in the summation of the terms – for example

$$S = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_{k,k-j} \right).$$

Since the sum may be calculated in “any order”, the convergence is called unconditional.
Notes on Unconditional Convergence

Let $A$ be an index set$^1$. For each $\alpha \in A$, let $z_\alpha$ be a complex number. Then the sum

$$\sum_{\alpha \in A} z_\alpha$$

converges unconditionally to $S$ if almost all the finite sums are close to $S$ in the precise sense:

Given $\epsilon > 0$, there is a finite set $F_\epsilon$ such that if $F$ is any finite subset of $A$, $F \supseteq F_\epsilon$, then

$$\left| \sum_{\alpha \in F} z_\alpha - S \right| < \epsilon.$$

Note that if $\sum z_\alpha$ converges unconditionally to $S$, then $\sum \Re z_\alpha$ converges unconditionally to $\Re S$ and $\sum \Im z_\alpha$ converges unconditionally to $\Im S$.

In particular, if $z_\alpha = x_\alpha \geq 0$, then

$$S = \sup_{F \text{ finite}} \sum_{\alpha \in F} x_\alpha.$$

In this case $\sum x_\alpha$ converges unconditionally iff the set of finite sums $\sum_{\alpha \in F} x_\alpha$, $F$ finite, is bounded.

Let $x_\alpha$ be real. Define

$$x_\alpha^+ = \max (x_\alpha, 0),$$
$$x_\alpha^- = \max (-x_\alpha, 0),$$

so that

$$x_\alpha = x_\alpha^+ - x_\alpha^-,$$
$$|x_\alpha| = x_\alpha^+ + x_\alpha^-.$$

$^1$ For technical reasons, if $A = \emptyset$, the empty set, interpret $\sum_{\alpha \in \emptyset} z_\alpha = 0$. 
Theorem. Let $x_\alpha$ be real. The sum $\sum_{\alpha \in A} x_\alpha$ converges unconditionally iff the sums $\sum_{\alpha \in A} x^+_\alpha$, $\sum_{\alpha \in A} x^-_\alpha$, and $\sum_{\alpha \in A} |x_\alpha|$ all converge unconditionally.

Proof. There is a finite set $F_1$ such that $\left| \sum_{\alpha \in F, F \supseteq F_1} x_\alpha - S \right| < 1$. Then if $F$ is any finite set,

$$\sum_{\alpha \in F} x^+_\alpha \leq \sum_{\alpha \in F \backslash F_1} x^+_\alpha + \sum_{\alpha \in F \cap F_1} x^+_\alpha \leq 1 + \sum_{\alpha \in F_1} |x_\alpha| \leq C.$$

Thus all the finite sums of $\sum_{\alpha \in A} x^+_\alpha$, $\sum_{\alpha \in A} x^-_\alpha$, and $\sum_{\alpha \in A} |x_\alpha|$ are bounded.

Summary

By considering $z_\alpha = x^+_\alpha - x^-_\alpha + i(y^+_\alpha - y^-_\alpha)$, we obtain:

1. The sum $\sum_{\alpha \in A} z_\alpha$ converges unconditionally iff

   - The sum $\sum_{\alpha \in A} |z_\alpha|$ converges unconditionally.

   iff

   - The set of finite sums $\sum_{\alpha \in F} |z_\alpha|$, $F$ finite, is bounded.

2. If $\{A_j\}$ is a sequence of sets increasing to $A$ ($A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots, A = \bigcup_{j=0}^{\infty} A_j$), it is easy to see that if $\sum_{\alpha \in A} z_\alpha$ converges unconditionally, then

   $$\sum_{\alpha \in A} z_\alpha = \lim_{j \to \infty} \sum_{\alpha \in A_j} z_\alpha.$$

3. Rearrangement: If $\sum_{\alpha \in A} z_\alpha$ converges unconditionally, and $A$ is a disjoint union of sets,
\( A = \bigcup_{j=0}^{\infty} B_j \), with the \( B_j \) pairwise disjoint, then

\[
\sum_{\alpha \in A} z_{\alpha} = \sum_{j=0}^{\infty} \left( \sum_{\alpha \in B_j} z_{\alpha} \right).
\]