M417

Fall 1996

hw11.tex due Nov 6, 1996

Let $f(z)$ have an isolated singularity at $z = 0$.

1. Suppose that $|f(z)|$ is bounded away from 0 for z near 0; *i. e.*, there are $R > 0$, $\delta > 0$, such that for $0 < |z| < R$, $f(z)$ is analytic and $|f(z)| > \delta$. Then f does not have an essential singularity at $z = 0$.

2. If f has an essential singularity at $z = 0$, and w_0 is any complex number, then there is a sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} f(z_n) = w_0$.

3. If $\lim_{z\to 0} |f(z)| = \infty$, then f has a pole at $z = 0$, but $f(z)$ does not have an essential singularity at $z = 0$.

Remark: One can now show the following result:

Theorem. *Suppose that $g(z)$ is an entire function such that $\lim_{z\to\infty} g(z) = \infty$. Then $g(z)$ is a polynomial.*

Outline of Proof: The function $f(z) = g\left(\frac{1}{z}\right)$ satisfies the hypotheses of the third exercise. For z near 0, $f(z) = \sum_{n=0}^{N} b_n z^{-n}$ and $g(z) = \sum_{n=0}^{N} b_n z^n$. Why are there no other nonzero terms?