M417

Fall 1996

hw6.tex due October 18, 1996

1. Suppose that $f(z)$ is an entire (analytic in the entire complex plane) function. Suppose that there is are constants A and B such that $|f(z)| \leq A + B\,|z|^N$. Prove that $f^{(N+1)}(z) = 0$. Conclude that $f(z)$ is a polynomial of degree $\leq N$.

2. Let $f(z)$ be analytic in the domain $D_R \equiv \{z|0 < |z| < R\}$. Show that for $0 < \epsilon < |z| < R$,
$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$
Both integrals are taken in the positive direction.

3. Let f be as in problem 2. Suppose that in addition f is bounded in the domain D_R. Show that for $0 < |z| < R$,
$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$
Conclude that $\lim_{z\to 0} f(z)$ exists.

N.B. The function $f(z)$ can be extended to be analytic in the domain $\{|z| \leq R\}$. This shows that an isolated singularity at which an analytic function remains bounded (in a deleted neighborhood) is *removable*.