Math 165: Revenue Streams and Mortgages

This file is `http://www.math.uic.edu/~jlewis/math165/165mortgage.pdf`

I. Income and Investment Streams

Simple Model

If a *present value* P is invested at time $t = 0$ with continuous compounding (CC) at rate r, the *future value* at time $t = T$ is

$$B = B(T) = Pe^{rT} = P(0)e^{rT}.$$

If we wish to have *future value* B at time $t = T$, we should invest a *present value* P given by

$$P = P(0) = Be^{-rT} = B(T)e^{-rT}.$$

If we wish to withdraw amounts $B_1, B_2, \ldots, B_N$, at times $T_1, T_2, \ldots, T_N$, we must have present value

$$P = P_1 + P_2 + \ldots + P_N = B_1 e^{-rT_1} + B_2 e^{-rT_2} + \ldots + B_N e^{-rT_N} = \sum B_i e^{-rT_i}.$$

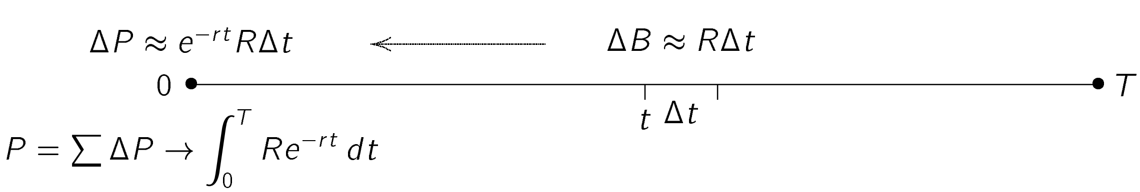
Continuous Model – Income Stream

We wish to withdraw a *continuous income stream* – to withdraw continuously at a rate R [dollars/year] for T [years].

At a typical time t, over a period Δt, we will withdraw $\Delta B \approx R\Delta t$, a future value at time t. Thus we need a present value (investment) $\Delta P \approx e^{-rt} R\Delta t$. The total present value needed is

$$P = \sum \Delta P \approx \sum_{t \text{ from } 0 \text{ to } T} e^{-rt} R\Delta t \approx \int_0^T Re^{-rt}\, dt.$$

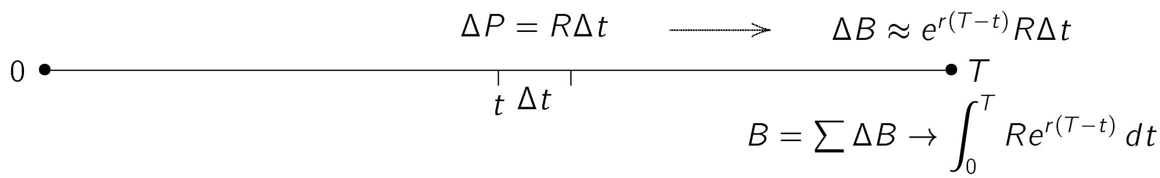
Figure 1.
Present Value P of an Income Stream R, $0 \le t \le T$



Similarly, if we *invest* continuously at rate R [dollars/year] for T [years], the *future value* B at time T will be given by

$$B = \int_0^T R\, e^{r(T-t)}\, dt.$$

Figure 2.
Future Value B of an Investment Stream R, $0 \le t \le T$



Inflation Adjustments

Similar arguments can be made if the rate, $R = R(t)$, depends on t. For example, the rate of contribution (investment) or income (revenue) might be continuously adjusted for inflation. In this case the formulas become:

If we wish to withdraw a *continuous income stream* – to withdraw continuously at a rate $R(t)$ [dollars/year] for T [years], we need a present value

$$P = \int_0^T R(t)\, e^{-rt}\, dt.$$

In particular if we make a *cost of living adjustment* (COLA), of $r_1\%$ annually,

$$R(t) = R_0 e^{r_1 t},$$
$$P = \int_0^T R_0\, e^{(r_1 - r)t}\, dt.$$

If we *invest* continuously at a rate $R(t)$ [dollars/year] for T [years], the *future value* B at time T will be given by

$$B = \int_0^T R(t)\, e^{r(T-t)}\, dt.$$

Pricing Annuities

For more on cost of living adjustments (COLA), see an exercise from Math 165 at UIC
`Math 165 Spring 2009 SA3: Pricing Annuities`
`http://www.math.uic.edu/~jlewis/math165/165sa309.pdf`

II. **Mortgage Payments**

A mortgage of P_0 for T years with an annual interest rate r is traditionally paid off at a fixed [annual] rate R.

If $P(t)$ is the principal remaining at time t, in a [small] time period Δt, the payment is $R\,\Delta t$, the interest accrued is (or $\approx$) is $rP(t)\,\Delta t$ so that the change in principal is

$$\Delta P \approx -R\,\Delta t + rP\,\Delta t.$$

Using the language of differentials,♣

$$dP = -R\,dt + rP\,dt.$$

Using the magic of differentials, we *divide by* dt♠ to obtain the differential equation

$$\frac{dP}{dt} = -R + rP.$$

There are many ways to find all solutions. Let $P(t) = u(t)e^{rt}$. Then

$$\begin{aligned} P' - rP &= u'e^{rt}, \\ u'e^{rt} &= -R, \\ u' &= -Re^{-rt}, \\ u(t) &= \frac{R}{r}e^{-rt} + C, \\ P(t) &= \frac{R}{r} + Ce^{rt}. \end{aligned}$$

In addition there are two *boundary conditions* satisfied by $P(t)$:

$$P(0) = P_0, P(T) = 0.$$

♣ In our case, $\approx$ means that the *error*, which depends on t and Δt, satisfies

$$\lim_{\Delta t \to 0} \frac{error(t, \Delta t)}{\Delta t} = 0.$$

♠ *divide by* dt means: replace dt by $\Delta t \neq 0$, divide by Δt, and let $\Delta t \to 0$.

Using the initial condition, $P(0) = P_0$,

$$C = P_0 - \frac{R}{r},$$
$$P(t) = \frac{R}{r} + \left(P_0 - \frac{R}{r}\right) e^{rt}$$

If T is given, solve the equation $P(T) = 0$ for R:

$$R = \frac{rP_0 e^{rT}}{e^{rT} - 1}$$

Since R is the annual rate of payment, the monthly payment is $M = R/12$.

If R is given, solve the equation $P(T) = 0$ for T:

$$0 = \frac{R}{r} + \left(P_0 - \frac{R}{r}\right) e^{rT}$$
$$e^{rT} = \frac{R}{R - rP_0}$$
$$T = \frac{1}{r} \ln\left(\frac{R}{R - rP_0}\right)$$

N.B. In a practical application, $R - rP_0 > 0$. The case $R - rP_0 = 0$ corresponds to paying interest only.

Investigations

1. **Variable Rate Mortgages.** At a time $t = T_1$, the rate changes from r to r_1. Investigate

- If the monthly payment M is held constant, how does the term (expiration date) of the loan change?
- If the term (expiration date) of the loan is unchanged, how does the monthly payment M change?

2. **Accelerated Payments.** At time $t = T_1$, make an extra payment R_1 and continue making the monthly payment M. How is the expiration date of the loan changed?

3. **Rules of Thumb?** Determine the veracity of the statements:

- Doubling the monthly payment M halves the period T of the mortgage.
- Doubling the period T of the mortgage halves the monthly payment M.

4. **Compare.** Several mortgage calculators are available on the web.

   Mortgage Calculator - Mortgage-calc.com
   `http://www.mortgage-calc.com/mortgage/simple.html`

   Take several common loans (for example \$100,000, 30 years at 5.75%,). Compare the monthly payment M you have calculated using continuous compounding (CC) and the monthly payment calculated by your favorite web mortgage calculator. Are the results different?

5. **Monthly Compounding.** In practice, interest is *not* compounded continuously(CC). A discrete calculation using monthly compounding is used. If an annual rate r is compounded M equally spaced times in a year, the *annual percentage rate* (APR) is

$$\text{APR} = \left(1 + \frac{r}{M}\right)^M - 1.$$

   The actual annual interest paid is the same as if the APR rate were *simply compounded.*

   On most consumer loans, the APR is recorded.

Discrete Payments

Mortgages and other loans are usually paid on a monthly basis. We will assume that there is a *nominal annual interest rate* r, and that payments of \$$R$ are made M times per year so that in a time period of $1/M$ years, for a loan of \$$P$, \$ $\left(1+\frac{r}{M}\right)P$ is due.

Start with a loan of P_0, and let P_n denote the principle remaining after the nth payment is made. The nth payment of R [dollars] covers the accrued interest ($\frac{r}{M}P_{n-1}$) and reduces the principal by $R-\frac{r}{M}P_{n-1}$.

There is a formula for P_n in terms of P_0, r, and M. It is convenient to let

$$\alpha = 1+\frac{r}{M}.$$

Then

$$\begin{aligned}
P_n &= \alpha P_{n-1} - R \\
&= \alpha\left(\alpha P_{n-2} - R\right) - R \\
&= \alpha^2 P_{n-2} - (\alpha+1)R \\
&= \alpha^2\left(\alpha P_{n-3} - R\right) - (\alpha+1)R \\
&= \alpha^3 P_{n-3} - \left(\alpha^2+\alpha+1\right)R \\
&= \ldots \\
&= \alpha^n P_0 - \left(\alpha^{n-1}+\ldots+\alpha+1\right)R. \\
&= \alpha^n P_0 - \frac{\alpha^n-1}{\alpha-1}R.
\end{aligned}$$

If the loan is paid off in T years with $N = MT$ payments, we have the equation for R.

$$\begin{aligned}
\alpha^N P_0 &= \left(\alpha^{N-1}+\ldots+\alpha+1\right)R \\
&= \frac{\alpha^N-1}{\alpha-1}R, \\
R &= (\alpha-1)\frac{\alpha^N}{\alpha^N-1}P_0.
\end{aligned}$$

Note that

$$\alpha - 1 = \frac{r}{M},$$

$$
\begin{aligned}
\alpha^N &= \left(1+\frac{r}{M}\right)^{MT} \\
&= \left(1+\frac{r}{M}\right)^{\frac{M}{r}rT} \\
&= \left[\left(1+\frac{r}{M}\right)^{\frac{M}{r}}\right]^{rT}.
\end{aligned}
$$

Let

$$
E = \left(1+\frac{r}{M}\right)^{\frac{M}{r}}.
$$

Note that $\frac{r}{M}$ is small and $\lim_{\epsilon\to 0}(1+\epsilon)^{\frac{1}{\epsilon}} = e$, so that $E \approx e$.

We obtain

$$
\begin{aligned}
R &= \frac{r}{M}\frac{E^{rT}}{E^{rT}-1}P_0, \\
E &= \left(1+\frac{r}{M}\right)^{\frac{M}{r}} \\
&\approx e.
\end{aligned}
$$

N.B. Since we have there are M payments in a year, the total *yearly payment*, Y, is

$$
Y = r\frac{E^{rT}}{E^{rT}-1}P_0. \tag{†}
$$

If Y and P_0 are given, solve the equation (†) for T:

$$
\begin{aligned}
\frac{Y}{rP_0} &= \frac{E^{rT}}{E^{rT}-1}, \\
\left(E^{rT}-1\right)\frac{Y}{rP_0} &= E^{rT}, \\
E^{rT} &= \frac{Y}{Y-rP_0}, \\
T &= \frac{1}{\ln(E)}\,\frac{1}{r}\ln\left(\frac{Y}{Y-rP_0}\right).
\end{aligned}
$$