Math 165 Parabolas

A typical problem about maximizing revenue

A manufacturer determines that when x hundred units of a particular commodity are produced, they can all be sold for a unit price given by the demand function $p = 80 - x$ dollars. What is the maximum revenue (in dollars)?

The demand function is *linear* – the quantity (demand) x and the price p are related by a linear relation. The problem is to set the price p (or demand x) so that the revenue R,

$$R = p \cdot x \text{ hundred dollars.}$$

is maximized.

Notice that

$$R = (80 - x)x$$

is a quadratic function of x; moreover, the coefficient of x^2 is negative, so the graph of $R(x)$ is a *parabola* opening downward. We even have that the parabola is presented in a *factored* form with roots at $x = 0$ and $x = 80$.

If we graph the parabola, it appears that R is maximized at the *vertex* of the parabola which occurs when $x = 40$, halfway between the roots!

We are already familiar with the *quadratic formula* for the roots of the equation

$$Ax^2 + Bx + C = 0$$

occur at

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

with the usual remarks about the case $B^2 - 4AC \leq 0$.

The *quadratic formula* is very much related to the process called *completing the square* – write

$$\begin{aligned} Ax^2 + Bx + C &= A\left(x^2 + (B/A)x + (C/A)\right) \\ &= A\left(\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right). \end{aligned}$$

It is apparent that the vertex of the parabola is located at

$$\begin{aligned} x &= -\frac{B}{2A}, \\ y &= A\left(\frac{C}{A} - \frac{B^2}{4A^2}\right) \\ &= \frac{4AC - B^2}{4A}. \end{aligned}$$

If $A > 0$, the graph opens upward and the quadratic function has a minimum value at the vertex.

If $A < 0$, the graph opens downward and the quadratic function has a maximum value at the vertex.

In our example $A = -1$, $B = 80$, $C = 0$. The revenue is maximized when

$$\begin{aligned} x &= -B/2A = 40(\text{ hundred units}), \\ p &= (80 - 40) \text{ dollars}, \\ R &= -\frac{80^2}{-4} = 1600(\text{ hundred dollars}) = \$160,000. \end{aligned}$$