

Algebra Symposium: Optimizing Fuel Consumption

To view animations:

<http://www2.math.uic.edu/~lewis/math165/asavgcost.htm>.

The Problem

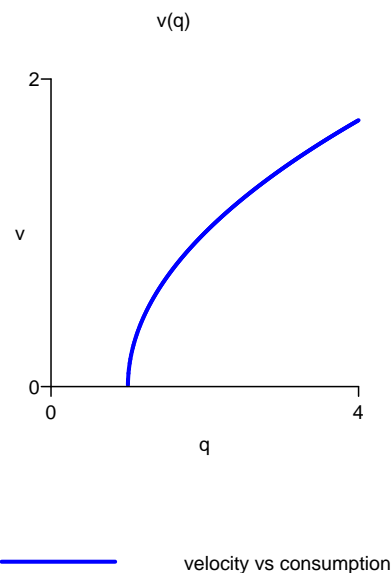
It is asserted that if you drive a car at constant speed; the number of miles per gallon will first increase and then decrease (or a reasonable range say 15 mph to 90 mph). Can someone explain the physics of this? Is there any sense that this function is quadratic?

An Approach Inspired by Economics¹

Graph velocity (speed) in mph (miles per hour) as a function of fuel consumption q in gph (gallons per hour).

Then the units of v/q are (miles/hour)/(gallons/hour) = (miles per gallon) = mpg!

Here is a typical graph of v versus g .

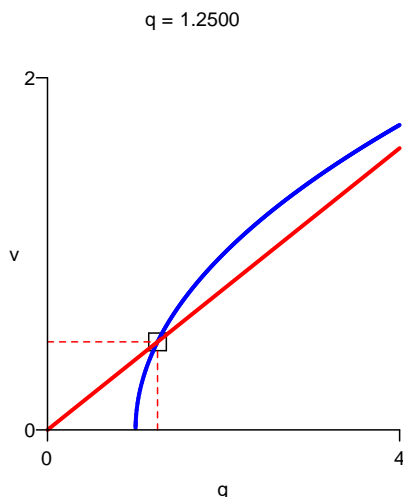


For a typical $v(q)$, the rate of change of v with respect to q , dv/dq , is decreasing – that is graph of $v(q)$ is concave down!² Observations on the importance of **concavity** are given in **Notes on the Approach** at the end of the note.

¹ This note was motivated by a lively discussion in the Department about the physics of the nature of the nonlinearity in the dependence of mpg on velocity. The “physics” assumed is that the functions which describe consumption, q , in terms of velocity, v , are positive, increasing, and concave upward.

² The graph is concave down if “at higher speeds equal changes in consumption will give smaller increases in speed.”

The slope of the line from (0, 0) to (q, v) is (v/q), the gas mileage (mpg) at speed v.
 Watch how the slope of the line changes as a function of q.



The (moving) box represents the point on the velocity vs consumption curve, the slope of the (moving) red line represents the mpg.

Notice that gas mileage is very poor (0 mpg!) when $v = 0$.

Notice that the slope of the red line is maximized when the line is tangent to the graph of $v(q)$; i.e. $\text{mpg} = dv/dq$.

The economic principle established is:³

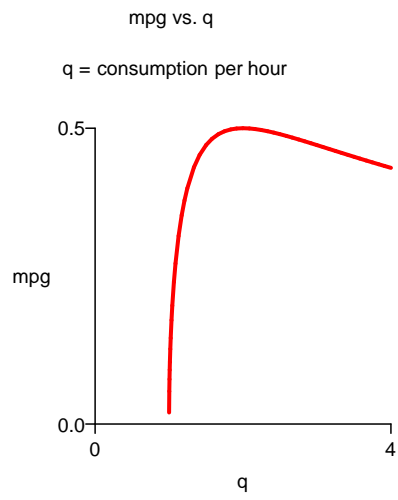
Criterion for Maximum Fuel Mileage. *Fuel efficiency is maximized when*

$$\frac{v}{q} = \frac{dv}{dq}.$$

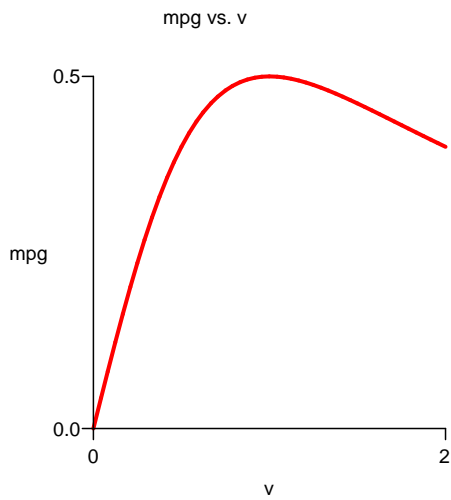
See the discussion [Notes on the Approach](#) at the end of the note.

³ See the discussion [Notes on the Approach](#) at the end of the note.

The Graph of Fuel Efficiency



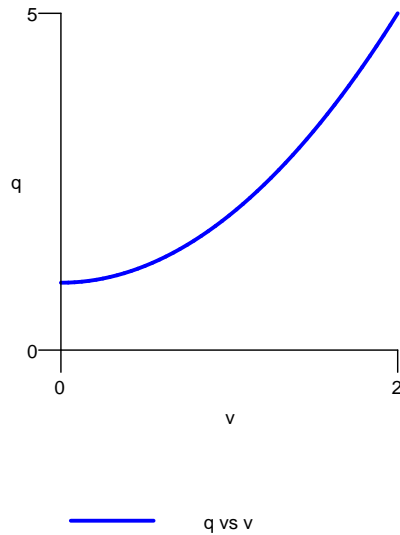
Now solve for v in terms of q .



A Direct Approach

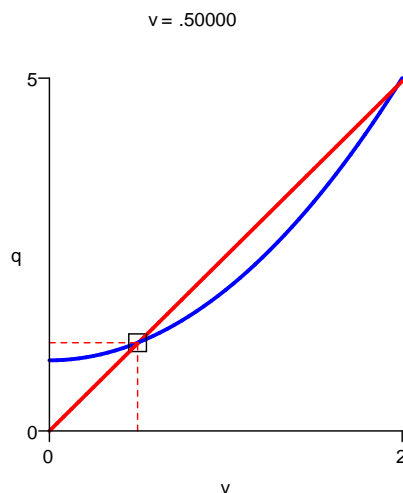
Graph the rate of consumption (in gph) as a function of the velocity v .

A typical graph of q as a function of v looks like this.



I am assuming that as v increases, more additional energy is required to increase v – the graph is concave up.⁴

The gas mileage (fuel efficiency) is measured by the ratio (v/q) , which is the *reciprocal of the slope of the line OP from the origin O to point $P = (v, q)$ on the graph.*

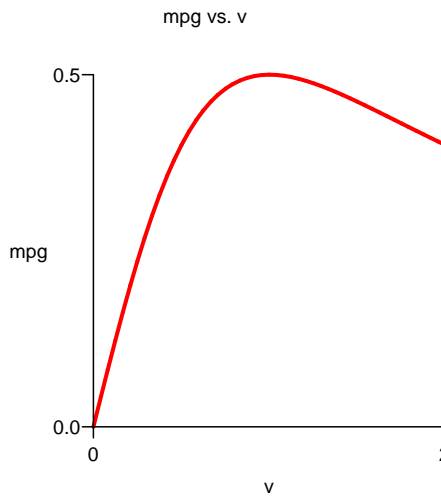


⁴ The graph is concave up if “at higher speeds equal changes in velocity require larger increases in consumption.”

Notice how the slope of OP changes as v changes. The slope of OP is minimized at the velocity v where the line OP is tangent to the graph! Since fuel efficiency is proportional to the reciprocal of the the slope of OP :

Criterion for Maximum Fuel Mileage. Fuel efficiency is maximized when $(q/v) = dq/dv$ (or $v/q = dv/dq$).

The plot of fuel efficiency is the graph of the function $v/q(v)$.



Notes on the Approach

1. **Mathematical Assumption – Concavity** This note was motivated by a lively discussion in the Department about the physics of the nature of the nonlinearity in the dependence of mpg on velocity. The key assumption is that the rate of fuel consumption, $q(v)$ is a nonnegative, increasing, concave upward function of the velocity, v .

See

<http://www2.math.uic.edu/~lewis/math165/asfuel.pdf>.

2. **Microeconomics – Elasticity.** Since the author (Jeff Lewis) is currently teaching Calculus for Business, we were reminded of a similar problem on Cost–Benefit Analysis in microeconomics.

The approach outlined here was motivated by

Marginal Analysis Criterion for Minimal Average Cost. Average cost is minimized at the level of production where average cost equals marginal cost; that is $A(q) = C'(q)$.

We showed

Criterion for Maximum Fuel Mileage. Fuel efficiency is maximized when $(q/v) = dq/dv$ (or $v/q = dv/dq$).

There is room for exploration. If we interpret the **Criterion for Maximum Fuel Mileage** in noncalculus terms – replacing (dv/v) and (dq/q) by small percentage changes in v and q , the economic principle seems to be that *fuel efficiency is optimized when small percentage changes in v produce approximately equal percentage changes in q* . Another characterization is that the *v elasticity of q is 1*.

See

<http://www2.math.uic.edu/~lewis/math165/math165avgcost.htm>.