Viewpoints

When we write

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n + ..., \\
&= \sum_{n=0}^{\infty} a_n x^n, \\
a_n &= \frac{f^{(n)}(0)}{n!} \\
P_N(x) &= P_{N,0}(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_N x^N, \\
&= \sum_{n=0}^{N} a_n x^n, \\
R_N(x) &\equiv f(x) - P_N(x),
\end{aligned}
$$

there are two contexts in which we are saying

$$f(x) \approx P_N(x).$$

The *local* context is that, fixing N,

$$f(x) = P_N(x) + \mathit{error},$$

where, as $x \to 0$ the *error* $[= R_N(x)]$ is much much smaller than x^N – the last and smallest term which has been included in P_N.

Thus, fixing N, $P_N(x)$ is a very good approximation for x near 0.

The *global* context is that for all x in an interval,

$$f(x) = P_N(x) + \mathit{error},$$

where, as $N \to \infty$, the *error* $[= R_N(x)]$ is small – the infinite series $\sum_{n=0}^{\infty} a_n x^n$ converges to the value $f(x)$.

Thus, fixing x [or varying x over a fixed interval], $P_N(x)$ is a very good approximation for N very large.

From the *local* viewpoint, since

$$P_0(x) = f(0),$$

the statement $f(x) \approx P_0(x)$ gives the statement $f(x) \approx f(0)$ for x near 0 – or $f(x)$ is continuous at $x = 0$.

Note that

$$P_1(x) = f(0) + f'(0) \cdot (x - 0)$$

gives the tangent line approximation to $f(x)$.

The *global* viewpoint is required to give an efficient way to calculate, e. g., e^x, for, say $-0.1 \leq x \leq 0.1$

For example, in treating continuous compounding of interest, we might need e^r, with $r = 6.1\% = .061$ or $r = 6.9\% = .069$. A very good approximation is

$$e^r \approx 1 + r + \frac{r^2}{2}, -0.1 \leq x \leq 0.1.$$