MthT 430 2005 Final Assessment

I. Definitions

1. (10 points) Define: The domain of a function f.

2. (10 points) Define (ϵ–δ): $\lim_{x \to a} f(x) = L$.

3. (10 points) Is the following a correct definition of continuity? Explain your answer.

Definition. *The function f is continuous at $x = a$ means: For some $\epsilon > 0$, there is a $\delta > 0$, such that, for all x, if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*

4. (10 points) Define: The number b is the *least upper bound* of a nonempty set of numbers A.

5. Assuming (P1) – (P12), State precisely property (P13) of the real numbers or one of its equivalent statements.

II. Examples

6. (10 points) Give an example of a function f with domain $[0, 1]$ such that

- f is continuous on $(0, 1]$,
- f is bounded above on $[0, 1]$, and
- f does not assume a maximum value on $[0, 1]$.

7. (10 points) Find the decimal and binary expansions of $x = \dfrac{1}{5}$.

8. (10 points) Give an example of two functions f and g such that $f \circ g$ and $g \circ f$ have **the same nonempty domains**, but $f \circ g \neq g \circ f$. Be sure to specify domain (f), domain (g), domain $(f \circ g) =$ domain $(g \circ f)$.

9. (10 points) Give an example of a function $f(x)$ defined for all real numbers such that, for all a, $\lim_{x \to a} f(x)$ does **not** exist.

10. (10 points) Give an example of a nonempty bounded set A_Q *of rational numbers* whose least upper

bound *is not a rational number*.

III. Proofs

11. (20 points) Show, using only P1 – P9:

$$-(ab) = (-a)\, b.$$

You may abbreviate (*distributive, P1, ...*).

12. (20 points) Show by mathematical induction or otherwise: For all natural numbers $n = 1, 2, \ldots$,

$$1 + 2 + \ldots + n = \frac{n\,(n+1)}{2}.$$

13. (20 points) Prove ($\epsilon - \delta$):

Theorem. *If*

$$\lim_{x \to a} f(x) = L \text{ and}$$
$$\lim_{x \to a} g(x) = M,$$

then

$$\lim_{x \to a} \left(f(x) + g(x) \right) = L + M.$$

14. (20 points) Let f be defined on $[0, 1)$ be such that

- f is increasing on $[0, 1)$ (If $0 \le x_1 < x_2 < 1$, then $f(x_1) < f(x_2)$.)
- f is bounded above on $[0, 1)$.

Prove that

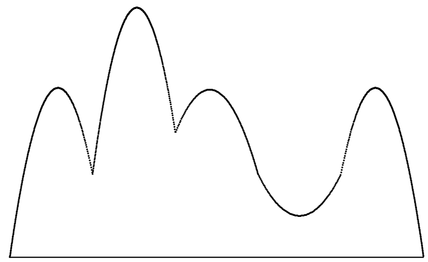
$$\lim_{x \to 1^-} f(x) = L$$

exists.

Hint: State precisely the version of (P13) that you use.

IV. County Line Theorem

15. (20 points) A county is bounded on the south by a horizontal line and bounded on the North by Meandering River.



The folks from the East and West of the County don't get along very well and want to to split the county into two parts of equal area. Show that it is possible to use a vertical line to divide the region into two regions of equal area.

V. Essay

16. (Letter Grade: A - E) Turn in an essay on a topic of your choice that is very relevant to the material considered in the course. Your essay should include at least one substantial example and at least one substantial theorem and its proof.