MthT 430 Term Project 2006

This project should be completed by a group of not less than two nor more than four persons. The group should turn in **one** typeup of the project. The paper should include a description of each member's contributions to the project.

The project should be **typed**. For suggestions on typing, see

`http://www2.math.uic.edu/~lewis/mtht430/430type.pdf`

Assignment due dates:

November 22, 2006 – 6 PM: Progress Report – A note to `jlewis@uic.edu` on your progress on the project – include the names of the members of your group.

November 29, 2006 – 5 PM: Completed typed project due.

I. Warmup – Inequalities Again and a Useful Fact

1. Show that if x and y are numbers, then $x \leq y$ if and only if for every $\epsilon > 0$, $x < y + \epsilon$.
2. Let A be a bounded set of numbers. Define
$$-A = \{-x \mid x \in A\}.$$
Show that
$$\inf A = -\sup(-A).$$

II. Understanding sup and inf – Equivalent Definitions

It is useful to note a working characterization of $\sup A$ $(A \neq \emptyset)$:

If $A \neq \emptyset$, $\sup A$ is a number α such that

$$\begin{cases} \text{For every } x \in A,\ x \leq \alpha, \text{ and} \\ \text{For every } \epsilon > 0, \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon. \end{cases}$$

The first condition means that *α is an upper bound for A*. The second condition means *for every $\epsilon > 0$, $\alpha - \epsilon$ is not an upper bound for A*.

3. Let A be a nonempty set of numbers which is bounded above. Show that

$$b = \sup A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x < b + \epsilon & \text{for all } x \in A, \text{ and} \\ x > b - \epsilon & \text{for some } x \in A. \end{cases}$$

4. Let A be a nonempty set of numbers which is bounded below. Show that

$$b = \inf A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x > b - \epsilon & \text{for all } x \in A, \text{ and} \\ x < b + \epsilon & \text{for some } x \in A. \end{cases}$$

III. Adding $\sup$ and $\inf$

5. (See Chapter 8 – Problem 13) Let A and B be two nonempty sets of numbers which are bounded (both above and below). Define

$$A + B = \{x \mid x = a + b, a \in A, b \in B\}.$$

Show that

$$\sup(A + B) = \sup A + \sup B.$$

Show that

$$\inf(A + B) = \inf A + \inf B.$$

IV. More Adding $\sup$ and $\inf$

If f is a bounded function on $[0, 1]$, we define

$$\sup f = \sup_{x\in[0,1]} f(x)$$
$$\inf f = \inf_{x\in[0,1]} f(x)$$

6. (Easy - see also Spivak Chapter 8 - Problem 13.) Show that if f and g are bounded functions on $[0, 1]$, then

$$\sup(f + g) \leq \sup f + \sup g.$$

7. Give an example of a pair of bounded functions f and g on $[0,1]$ such that

$$\sup(f+g) < \sup f + \sup g.$$

8. Show that if f and g are bounded functions on $[0,1]$, then

$$\inf f + \inf g \le \inf(f+g).$$

9. Show that if f and g are bounded functions on $[0,1]$, then

$$\inf f + \sup g \le \sup(f+g).$$

10. (Easy – See the previous problem(s).) and Show that if f and g are bounded functions on $[0,1]$, then

$$\inf(f+g) \le \inf f + \sup g.$$

11. (Monster Counterexample to Equality) Your group has shown: For two bounded functions f and g on $[0,1]$,

$$\begin{aligned} \inf f + \inf g &\le \inf(f+g) \\ &\le \inf f + \sup g \\ &\le \sup(f+g) \\ &\le \sup f + \sup g. \end{aligned}$$

Give an example of a pair of bounded functions f and g on $[0,1]$ such that

$$\begin{aligned} \inf f + \inf g &< \inf(f+g) \\ &< \inf f + \sup g \\ &< \sup(f+g) \\ &< \sup f + \sup g. \end{aligned}$$

N.B. The string of inequalities is related to similar inequalities regarding $\limsup$ and $\liminf$ in

`http://www2.math.uic.edu/~lewis/mtht430/chap8fproj.pdf`