MthT 430 Term Project 2006 Notes Revised

Problem 9 Remarks Revised December 6, 2006

This project should be completed by a group of not less than two nor more than four persons. The group should turn in **one** typeup of the project. The paper should include a description of each member's contributions to the project.

The project should be **typed**. For suggestions on typing, see

`http://www2.math.uic.edu/~lewis/mtht430/430type.pdf`

Assignment due dates:

November 22, 2006 – 6 PM: Progress Report – A note to `jlewis@uic.edu` on your progress on the project – include the names of the members of your group.

November 29, 2006 – 5 PM: Completed typed project due.

I. Warmup – Inequalities Again and a Useful Fact

1. Show that if x and y are numbers, then $x \leq y$ if and only if for every $\epsilon > 0$, $x < y + \epsilon$.
2. Let A be a bounded set of numbers. Define

$$-A = \{-x \mid x \in A\}.$$

Show that

$$\inf A = -\sup(-A).$$

II. Understanding sup and inf – Equivalent Definitions

It is useful to note a working characterization of $\sup A$ ($A \neq \emptyset$):

If $A \neq \emptyset$, $\sup A$ is a number α such that

$$\begin{cases} \text{For every } x \in A,\ x \leq \alpha, \text{ and} \\ \text{For every } \epsilon > 0, \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon. \end{cases}$$

The first condition means that α *is an upper bound for* A. The second condition means *for every* $\epsilon > 0$, $\alpha - \epsilon$ *is not an upper bound for* A.

3. Let A be a nonempty set of numbers which is bounded above. Show that

$$b = \sup A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x < b + \epsilon & \text{for all } x \in A, \text{ and} \\ x > b - \epsilon & \text{for some } x \in A. \end{cases}$$

4. Let A be a nonempty set of numbers which is bounded below. Show that

$$b = \inf A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x > b - \epsilon & \text{for all } x \in A, \text{ and} \\ x < b + \epsilon & \text{for some } x \in A. \end{cases}$$

III. Adding $\sup$ **and** $\inf$

5. (See Chapter 8 – Problem 13) Let A and B be two nonempty sets of numbers which are bounded (both above a nd below). Define

$$A + B = \{x \mid x = a + b, a \in A, b \in B\}.$$

Show that

$$\sup (A + B) = \sup A + \sup B.$$

Solution. If $x = a + b$, $a \in A$, $b \in B$, then

$$\begin{aligned} x &= a + b \\ &\leq \sup A + \sup B \end{aligned}.$$

Thus $\sup A + \sup B$ is an upper bound for $A + B$ and

$$\sup (A + B) \leq \sup A + \sup B.$$

Given $\epsilon > 0$, there is an $a \in A$ such that $a > \sup A - \epsilon$, and a $b \in B$ such that $b > \sup B - \epsilon$. Then

$$\begin{aligned} \sup (A + B) &\geq a + b \\ &\geq \sup A + \sup B - 2\epsilon \end{aligned},$$

so that

$$\sup A + \sup B < \sup (A + B) + 2\epsilon.$$

Thus, using Problem 1,

$$\sup A + \sup B \leq \sup (A + B).$$

Show that

$$\inf (A + B) = \inf A + \inf B.$$

IV. More Adding sup and inf

If f is a bounded function on $[0,1]$, we define

$$\sup f = \sup_{x\in[0,1]} f(x)$$
$$\inf f = \inf_{x\in[0,1]} f(x)$$

6. (Easy - see also Spivak Chapter 8 - Problem 13.) Show that if f and g are bounded functions on $[0,1]$, then
$$\sup (f+g) \le \sup f + \sup g.$$

Solution. The number $\sup f + \sup g$ is an upper bound for the set $\{(f+g)(x) \,|x \in [0,1]\}$.

7. Give an example of a pair of bounded functions f and g on $[0,1]$ such that
$$\sup (f+g) < \sup f + \sup g.$$

8. Show that if f and g are bounded functions on $[0,1]$, then
$$\inf f + \inf g \le \inf (f+g).$$

9. Show that if f and g are bounded functions on $[0,1]$, then
$$\inf f + \sup g \le \sup (f+g).$$

Solution 1. Fix $x \in [0,1]$. Then

$$\begin{aligned} g(x) &= (f+g)(x) - f(x) \\ &\le \sup (f+g) - \inf f. \end{aligned}$$

and
$$\sup g \le \sup (f+g) - \inf f.$$

Solution 2. Given $\epsilon > 0$, there is an $x \in [0,1]$ such that $g(x) > \sup g - \epsilon$. For this x,

$$\begin{aligned} \inf f &\le f(x) \\ &= (f+g)(x) - g(x) \\ &\le \sup (f+g) - \sup g + \epsilon. \end{aligned}$$

It follows that: for every $\epsilon > 0$,

$$\inf f + \sup g \leq \sup (f+g) + \epsilon.$$

Remark. Several groups tried to use the two [in]equalities

$$\inf f \leq \sup f, \tag{* – true}$$
$$\sup f + \sup g = \sup (f+g)\,. \tag{** – false}$$

[In]equality (**) looks like Problem 5, but was shown not always true by the counterexample in Problem 7. I think the confusion arises from the interpretation of the notation

$$\begin{aligned}\sup f &= \sup_{x\in[0,1]} f(x) \\ &= \sup \{ f(x) \,|x \in [0,1] \}\,.\end{aligned}$$

Note that, as sets,

$$\{(f+g)(x) \,|x \in [0,1] \} \subseteq \{ f(x) \,|x \in [0,1] \} + \{ g(x) \,|x \in [0,1] \},$$

but equality is not always true. The right hand side is

$$\{ f(x) \,|x \in [0,1] \} + \{ g(x) \,|x \in [0,1] \} = \{ f(x) + g(y) \,|x \in [0,1]\,, y \in [0,1] \}\,.$$

Remark on Language. Note that

$$\begin{aligned}\{ g(x) \,|x \in [0,1] \} &= \{ g(y) \,|y \in [0,1] \} \\ &= \{ g(t) \,|t \in [0,1] \} \\ &= \{ g(\Diamond) \,|\Diamond \in [0,1] \} \\ &= \ldots.\end{aligned}$$

Name an internal variable – $x, y, t, \Diamond, \ldots$ – and then say what it means. Similarly, with a given function f,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{y\to a} f(y) \\ &= \lim_{t\to a} f(t) \\ &= \lim_{\Diamond\to a} f(\Diamond) \\ &= \ldots.\end{aligned}$$

10. (Easy – See the previous problem(s).) and Show that if f and g are bounded functions on $[0, 1]$, then

$$\inf\,(f + g) \leq \inf f + \sup g.$$

11. (Monster Counterexample to Equality) Your group has shown: For two bounded functions f and g on $[0, 1]$,

$$\begin{aligned}\inf f + \inf g &\leq \inf\,(f + g) \\ &\leq \inf f + \sup g \\ &\leq \sup\,(f + g) \\ &\leq \sup f + \sup g.\end{aligned}$$

Give an example of a pair of bounded functions f and g on $[0, 1]$ such that

$$\begin{aligned}\inf f + \inf g &< \inf\,(f + g) \\ &< \inf f + \sup g \\ &< \sup\,(f + g) \\ &< \sup f + \sup g.\end{aligned}$$

Monster Counterexample. Let

$$f(x) = \begin{cases} 2, & 0 \leq x < 0.5, \\ 0, & 0.5 \leq x \leq 1. \end{cases}$$

$$g(x) = \begin{cases} 0, & 0 \leq x < 0.25, \\ 1, & 0.25 \leq x \leq 0.75, \\ 2, & 0.75 < x \leq 1. \end{cases}$$

$$f(x) + g(x) = \begin{cases} 2, & 0 \leq x < 0.25, \\ 3, & 0.25 \leq x < 0.5, \\ 1, & 0.5 \leq x \leq 0.75, \\ 2, & 0.75 < x \leq 1. \end{cases}$$

$$\begin{aligned}\inf f = \inf g &= 0, \\ \inf\,(f + g) &= 1, \\ \sup\,(f + g) &= 3, \\ \sup f = \sup g &= 2.\end{aligned}$$

N.B. The string of inequalities is related to similar inequalities regarding lim sup and lim inf in

`http://www2.math.uic.edu/~lewis/mtht430/chap8fproj.pdf`