MthT 430 sqrt (2) is Irrational

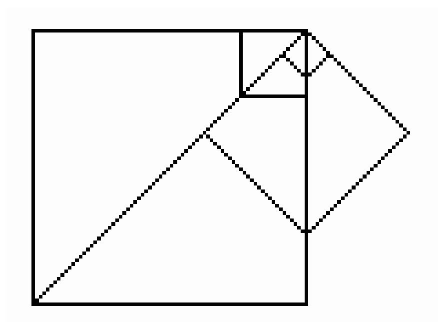
From:

http://www.cut-the-knot.org/proofs/sq_root.shtml

Square root of 2 is irrational

Proof 6 (Statement and figure of A. Bogomolny)

If $x = 2^{1/2}$ were rational, there would exist a quantity s commensurable both with 1 and x: $1 = sn$ and $x = sm$. (It's the same as assuming that $x = m/n$ and taking $s = 1/n$.) The same will be true of their difference $x - 1$, which is smaller than x. And the process could continue indefinitely in contradiction with the existence of a minimal element. The game Euclid might have played always ends!



JL Explanation

Definition.[1] **commensurable**: *Mathematics* Exactly divisible by the same unit an integral number of times. Used of two quantities.

If $x = \sqrt{2} = m/n$, m, n integers, after choosing $s = 1/n$, we have an isosceles right triangle with sides $1 = n\,s$ and hypotenuse $x = m\,s$. Now follow the Apostol program to construct an isosceles right triangle with sides $(m - n)\,s = x - 1$ and hypotenuse $t\,s$, where

$$\begin{aligned} \frac{t\,s}{x-1} &= \frac{m\,s}{n\,s}, \\ \frac{t}{m-n} &= \frac{m}{n}, \\ t &= \frac{1}{n}\left(m^2 - n\,m\right) \\ &= 2n - m. \end{aligned}$$

Continuing we get a sequence of isosceles right triangles with sides $n_j\,s$ and hypotenuse $m_j\,s$; after at most n steps we reach a contradiction.

[1] http://www.thefreedictionary.com/commensurable

Another JL Explanation

If $x = \sqrt{2} = m/n$, after choosing $s = 1/n$, rescale so that we have an isosceles right triangle with integer sides n and integer hypotenuse m. Now follow the Apostol program to construct an isosceles right triangle with integer sides $m - n$ and hypotenuse t, where

$$\begin{aligned} \frac{t}{m-n} &= \frac{m}{n}, \\ t &= \frac{1}{n}\left(m^2 - nm\right) \\ &= 2n - m. \end{aligned}$$

Continuing we get a sequence of isosceles right triangles with integer sides and integer hypotenuse; after at most n steps we reach a contradiction.