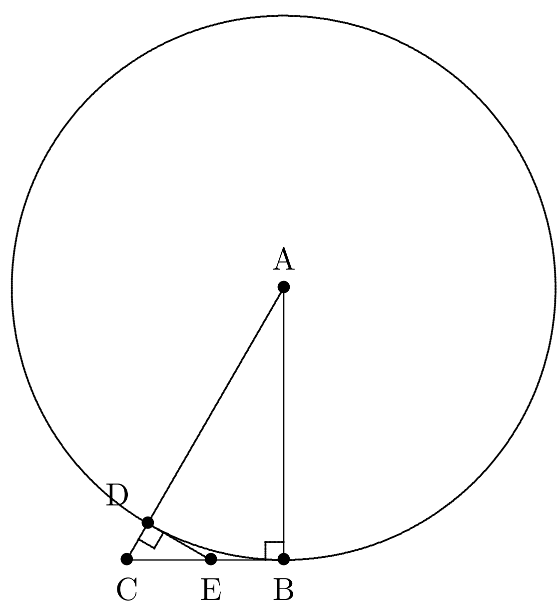
MthT 430 Apostol's Irrationality – More

Apostol remarks that a similar argument shows that $\sqrt{N^2+1}$ and $\sqrt{N^2-1}$ are irrational except in the obvious cases.



In the above picture note that $\Delta EDC \simeq \Delta ABC$, so that

$$\frac{CD}{CB} = \frac{DE}{AB} = \frac{CE}{AC}.$$

Irrationality of $\sqrt{N^2+1}$

Suppose that $\sqrt{N^2+1}$ is a rational number.

If $q^2\left(N^2+1\right) = p^2$ for natural numbers $q > 1$, $p > 1$, we may construct ΔABC with integer sides so that

$$\begin{aligned} AC &= q\sqrt{N^2+1}, \\ AB &= qN, \\ CB &= q. \end{aligned}$$

Choose the *smallest such* q. Then

$$CD = q\left(\sqrt{N^2+1} - N\right), \text{ an integer.}$$

Define

$$\alpha = \frac{CD}{CB} = \left(\sqrt{N^2+1} - N\right).$$

Then

$$\begin{aligned} CD &= \alpha\, CB = \alpha\, q = q',\ \text{an integer}, \\ DE &= \alpha\, AB = \alpha\, q\, N = q'\, N,\ \text{an integer}, \\ CE &= BC - BE = BC - DE \\ &= \alpha\, AC = q'\, \sqrt{N^2+1},\ \text{an integer}. \end{aligned}$$

Irrationality of $\sqrt{N^2-1}$

If $q^2\left(N^2-1\right) = p^2$ for natural numbers $q > 1$, p, we may construct ΔABC with integer sides so that

$$\begin{aligned} AB &= q\sqrt{N^2-1}, \\ AC &= qN, \\ CB &= q. \\ CD &= q\left(N - \sqrt{N^2-1}\right),\ \text{an integer}, \\ \frac{CD}{CB} &= \left(N - \sqrt{N^2-1}\right). \\ &= \beta. \end{aligned}$$

Then

$$\begin{aligned} CD &= \beta\, CB = \beta\, q = q',\ \text{an integer}, \\ DE &= \beta\, AB = \beta\, q\, \sqrt{N^2-1} = q'\, \sqrt{N^2-1},\ \text{an integer}, \\ CE &= \beta\, AC = \beta\, qN = q'\, N,\ \text{an integer}. \end{aligned}$$

`Tom M. Apostol, Irrationality of The Square Root of Two -- A Geometric Proof, American Mathematical Monthly` **107**`, No. 9 (Nov., 2000), pp. 841-842.`