MthT 430 Notes Chapter 01 Basic Properties of Numbers Replaced by Chap01

For the time being we will denote the numbers we are used to by **Numbers**. We shall list the primitive properties – that is, develop a minimal list of properties from which results can be deduced.

We shall assume there is a set **Numbers**, with binary operations + (plus, addition) and $\cdot$ (times, multiplication) defined.

Properties of + (Addition)

P1 For all a, b, c,

$$a + (b + c) = (a + b) + c$$

P2 There is a number 0 such that for all a,

$$a + 0 = 0 + a = a.$$

P3 For all a, there is a number $-a$ such that

$$a + (-a) = (-a) + a = 0.$$

P4 For all a, b,

$$a + b = b + a.$$

Properties of $\cdot$ (Multiplication)

P5 For all a, b, c,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

P6 There is a number $1 \neq 0$ such that for all a,

$$a \cdot 1 = 1 \cdot a = a.$$

P7 For all $a \neq 0$, there is a number a^{-1} such that

$$a \cdot (a^{-1}) = (a^{-1}) \cdot a = 1.$$

P8 For all a, b,

$$a \cdot b = b \cdot a.$$

Property of $\cdot$ with + (Glue)

P9 For all a, b, c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) = a \cdot b + a \cdot c.$$

Consequences of P1–P9

Theorem. Assuming P1 − P9, for all a, $a \cdot 0 = 0$.

The proof is given on page 7 of Spivak's book.

Proof.

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) \\ &= a \cdot 0 + a \cdot 0. && \text{(P9)} \end{aligned}$$

Now subtract $a \cdot 0$ from both sides of the equation

$$\begin{aligned} a \cdot 0 &= a \cdot 0 + a \cdot 0. && \text{(P9)} \\ a \cdot 0 + (-(a \cdot 0)) &= a \cdot 0 + a \cdot 0 - a \cdot 0 \\ 0 &= a \cdot 0. && \text{(P3, P1, P2, P3)} \end{aligned}$$

Theorem. *If*

$$a \cdot b = 0,$$

then either

$$a = 0 \text{ or } b = 0.$$

Proof. If $a \neq 0$ and $a \cdot b = 0$, then

$$\begin{aligned} a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot 0, \\ &= 0, && \text{Above} \\ \left(a^{-1} \cdot a\right) \cdot b &= 0, && \text{P1} \\ 1 \cdot b &= 0, && \text{P7} \\ b &= 0. && \text{P6} \end{aligned}$$

Similarly, if $b \neq 0$, then $a = 0$.

Positive Numbers and Order

We introduce the collection of positive numbers, P, and state the order properties in terms of P.

P10 For every a, one and only one of the following holds:

(i) $a = 0$,

(ii) a is in the collection P,

(iii) a is in the collection P.

P11 If a and b are in the collection P, then $a + b$ is in the collection P.

P12 If a and b are in the collection P, then $a \cdot b$ is in the collection P.

Sometimes we will refer to the collection of numbers satisfying P1–P9 as a *field*, and a collection of numbers satisfying P1–P12 as a *ordered field*. If we wish to emphasize the particular binary operations $+$, $\cdot$, and the set of positive numbers P, we will write

$$(\mathbf{Numbers}, +, \cdot, P)\,.$$

In turn , inequalities are defined in terms of of the positive numbers P. We say that $a > 0$ or $0 < a$ if a is in P.

Absolute value

Definition. The absolute value of a number a is defined as

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Properties of absolute value often involve a *proof by cases*.

Theorem. *For all a,*

$$-|a| \leq a \leq |a|\,.$$

Proof. If $a \geq 0$, then $|a| = a$, and

$$-|a| \leq 0 \leq a = |a|\,.$$

If $a \leq 0$, then $-|a| = a$, and

$$-|a| = a \leq 0 \leq |a|\,.$$

Theorem. (The Triangle Inequality) *For all numbers a, b,*

$$|a + b| \leq |a| + |b|\,.$$

Proof. We have

$$-|a| \leq a \leq |a|\,,$$
$$-|b| \leq b \leq |b|\,,$$

so that

$$-(|a|+|b|) \leq a+b \leq |a|+|b|.$$

Now break into the two cases $a+b \geq 0$ and $a+b \leq 0$.

- If $a+b \geq 0$, then $|a+b| = a+b \leq |a|+|b|$.
- If $a+b \leq 0$, then $|a+b| = -(a+b) \leq |a|+|b|$.

Remarks

Properties P1 – P4 say that the **Numbers** under addition form a *commutative group*.

Properties P5 – P8 say that the nonzero **Numbers** under multiplication form a *commutative group*.

The *distributive property*, P9, is the *glue* which makes the operations of addition and multiplication work together. Properties P1 – P8 alone, without P9, can lead to some weird structures.

See

`http://www.math.uic.edu/~jlewis/mtht430/491dist.pdf`

In words:

Addition

P1 Addition is associative

P2 Addition is commutative

P3 Zero

P4 Additive inverse

Multiplication

P5 Multiplication is associative

P6 Multiplication is commutative

P7 One (not Zero!)

P8 Multiplicative inverse

Plus and Times

P9 Multiplication distributes over addition

Order

P10 Trichotomy

P11 Sum of positives is positive

P12 Product of positives is positive