MthT 430 Notes Chapter 11 Significance of the Derivative

Maximum Point on a set A

Definition. *Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a* **maximum point for f on A** *if*

$$f(x) \geq f(y) \text{ for every } y \text{ in } A.$$

The number $f(x)$ itself is called the **maximum value of f on A**.

N.B. Several texts are inconsistent in distinguishing the value, x, the value of the function, $f(x)$, and the point, $(x, f(x))$, on the graph.

The basic relation between **the maximum point for f on an open interval** and the derivative is given in Theorem 1.

Theorem 1. *Let f be any function defined on (a, b). If x is a maximum point for f on (a, b), and f is differentiable at x, then $f'(x) = 0$.*

Definition. *Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a* **local [relative] maximum point for f on A** *if there is some $\delta > 0$ such that x is a maximum point for f on $A \cap (x - \delta, x + \delta)$.*

$$f(x) \geq f(y) \text{ for every } y \text{ in } A \cap (x - \delta, x + \delta).$$

A less technical statement is that $f(x) \geq f(y)$ for all nearby points y in A.

Definition. *A* **critical point** *of a function f is a number x such that*

$$f'(x) = 0.$$

The number $f(x)$ is called a **critical value** *of f.*

N.B. Once again there is often inconsistency in referring to x, $f(x)$, and the point $(x, f(x))$ on the graph.

To locate the maximum point of f on a closed interval $[a, b]$, we need only look at

- critical points of f in $[a, b]$ (usually a small number),
- end points a and b,
- points x in $[a, b]$ such that f is not differentiable (which should be obvious).

Rolle's Theorem. *If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then there is an x in (a,b) such that $f'(x)=0$.*

Proof. If f is constant on $[a,b]$, then $f'(x)=0$ for all x in (a,b). If f is not constant on $[a,b]$, then there is a maximum point or minimum point x for f on (a,b). At such a point, by Theorem 1, $f'(x)=0$.

Applying Rolle's Theorem to various functions, we obtain several important results.

Mean Value Theorem. *If f is continuous on $[a,b]$ and differentiable on (a,b), , then there is an x in (a,b) such that*

$$f'(x) = \frac{f(b)-f(a)}{b-a}.$$

.

Proof. Let

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a),$$

the secant line through $(a, f(a))$ and $(b, f(b))$. Then

$$F(x) = f(x) - g(x)$$

satisfies the hypotheses of Rolle's Theorem. There is an x in (a,b) such that

$$\begin{aligned} F'(x) &= f'(x) - \frac{f(b)-f(a)}{b-a} \\ &= 0. \end{aligned}$$

We could not resist the proof of a version of L'Hôpital's Rule.

Cauchy's Mean Value Theorem. *If f and g are continuous on $[a,b]$, and differentiable on (a,b), then there is an x in (a,b) such that*

$$[f(b)-f(a)]\,g'(x) = [g(b)-g(a)]\,f'(x).$$

.

Proof. Apply Rolle's Theorem to

$$H(x) = [f(b)-f(a)]\,(g(x)-g(a)) - [g(b)-g(a)]\,(f(x)-f(a)).$$

Theorem 9 (L'ÔPITAL'S RULE). *Suppose that*

$$\lim_{x\to a} f(x) = 0, \text{ and } \lim_{x\to a} g(x) = 0.$$

Suppose that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists.} \tag{$*$}$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{ exists,}$$

and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. Without loss of generality, assume that $f(a) = g(a) = 0$. Using (*), notice that there is a $\delta > 0$ such that, for $0 < |x - a| < \delta$, $g'(x) \neq 0$ for $0 < |x - a| < \delta$. By the Mean Value Theorem, for $0 < |x - a| < \delta$, $g(x) \neq 0$. Fix x. By the Cauchy Mean Value Theorem, there is a c_x (which depends on x) between a and x such that

$$f(x)g'(c_x) = g(x)f'(c_x),$$

$$f(x)g'(c_x) = g(x)f'(c_x) \tag{$\dagger$}$$

Dividing (†) by $g(x)$ ($\neq 0!$) and $g'(c_x)$, we obtain

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}.$$

As $x \to a$, $c_x \to a$, so that

$$\begin{aligned} \lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f'(c_x)}{g'(c_x)}. \\ &= \lim_{x\to a} \frac{f'(x)}{g'(x)}. \end{aligned}$$