A function $f$ is *convex* on an interval $I$ if every secant line is above the graph on $I$.

Algebraically, convexity is expressed by: If $X_1 < X_2$, then for $X_1 < X < X_2$,

$$f(X) \leq f(X_1) + \frac{f(X_2) - f(X_1)}{X_2 - X_1}(X - X_1).$$

1. Show that if $X_1 < X_2 < X_3$, then

$$\frac{f(X_2) - f(X_1)}{X_2 - X_1} \leq \frac{f(X_3) - f(X_1)}{X_3 - X_1},$$

$$\frac{f(X_3) - f(X_1)}{X_3 - X_1} \leq \frac{f(X_3) - f(X_2)}{X_3 - X_2}.$$  

This is an algebraic verification of the geometric observation that $\text{slope } \overline{P_1P_2} \leq \text{slope } \overline{P_1P_3}$ and $\text{slope } \overline{P_1P_3} \leq \text{slope } \overline{P_2P_3}$.
In particular, as $X_3$ decreases to $X_2$, the difference quotient
\[ \frac{f(X_3) - f(X_2)}{X_3 - X_2} \]
decreases and is bounded below by
\[ \frac{f(X_2) - f(X_1)}{X_2 - X_1}. \]
Thus
\[ \lim_{X \to X^+} \frac{f(X) - f(X_2)}{X - X_2} = D_+ f(X_2) \]
exists. The right hand limit of the difference quotient,
\[ D_+ f(x) = \lim_{\Delta x \to 0^+} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]
is called the right derivative or right derivative of $f$ at $x$.

Similarly the left derivative
\[ \lim_{X \to X^-} \frac{f(X) - f(X_2)}{X - X_2} = D_- f(X_2) \]
exists (and is $\leq D_+ f(X_2)$).

Thus we have shown: If $f$ is convex on an open interval $I = (a, b)$, then for each $x \in I$, the left derivative $D_- f(x)$ and the right derivative $D_+ f(x)$ both exist. If they are the same, then $f'(x_0)$ exists. Note that if $a < x_1 < x_2 < b$, then
\[ D_- f(x_1) \leq D_+ f(x_1) \leq D_- f(x_2) \leq D_+ f(x_2). \]
It follows that $D_- f(x)$ and $D_+ f(x)$ are nondecreasing functions on $I$.

2. Is the following result true? If not, give a counterexample.

**Theorem.** If $f$ is convex on an open interval $I = (a, b)$, then $f$ is continuous on $I$. 