MthT 430 Projects Chapter 1a Inequalities

Chapter 1, Problems 20, 21 and 22 in Spivak

20. Prove that if

$$|x - x_0| < \epsilon/2 \text{ and } |y - y_0| < \epsilon/2,$$

then

$$|(x + y) - (x_0 + y_0)| < \epsilon.$$

Proof: The idea is to express or estimate the expression in the *conclusion*,

$$|(x + y) - (x_0 + y_0)|\,,$$

in terms of expressions in the *hypothesis*,

$$|x - x_0| \text{ and } |y - y_0|\,.$$

$$\begin{aligned} |(x + y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\le |(x - x_0)| + |(y - y_0)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

21. Prove that if

$$\begin{aligned} |x - x_0| &< \min\left(\frac{\epsilon}{2\,(|y_0| + 1)}, 1\right), \\ |y - y_0| &< \min\left(\frac{\epsilon}{2\,(|x_0| + 1)}, 1\right), \end{aligned}$$

then

$$|x\,y - x_0\,y_0| < \epsilon.$$

Proof: The idea again is to express or estimate the expression in the *conclusion*,

$$|x\,y - x_0\,y_0|\,,$$

in terms of expressions in the *hypothesis*,

$$|x - x_0| \text{ and } |y - y_0|\,.$$

$$\begin{aligned}|x\,y - x_0\,y_0| &= |x\,(y-y_0) + (x-x_0)\,y_0| \\ &\le |x|\,|y-y_0| + |y_0|\,|x-x_0| \\ &= I + II.\end{aligned}$$

$$\begin{aligned} I &\le |(x-x_0)+x_0|\,|y-y_0| \\ &\le (1+|x_0|)\,\frac{\epsilon}{2\,(|x_0|+1)} \\ &< \epsilon/2. \\ II &\le |y_0|\,\frac{\epsilon}{2\,(|y_0|+1)} \\ &< \epsilon/2.\end{aligned}$$

22. Show that if $x_0 \neq 0$, and

$$|x - x_0| < \min\left(\frac{|x_0|^2\,\epsilon}{2}, \frac{|x_0|}{2}\right),$$

then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon.$$

Proof: Note that

$$\begin{aligned}|x| &= |x_0 + (x - x_0)| \\ &\ge |x_0| - \frac{|x_0|}{2} \\ &= \frac{|x_0|}{2}.\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{x_0}\right| &= \left|\frac{x - x_0}{x x_0}\right| \\ &= \frac{1}{|x|}\,\frac{1}{|x_0|}\,|x - x_0| \\ &\le \frac{2}{|x_0|}\,\frac{1}{|x_0|}\,|x - x_0| \\ &< \frac{2}{|x_0|}\,\frac{1}{|x_0|}\,\frac{|x_0|^2\,\epsilon}{2} \\ &= \epsilon.\end{aligned}$$