MthT 430 Chapter 2a Numbers of Various Sorts

Natural numbers N

We denote by **N** the set of *counting numbers*

$$1, 2, 3, \ldots .$$

The natural numbers (counting numbers, positive integers) **N** satisfy the *Principle of Mathematical Induction*: (PMI):

If A is a subset of **N** such that

$$\begin{cases} 1 \in A, \\ \text{Whenever } k \in A,\ k+1 \in A, \end{cases}$$

then

$$A = \mathbf{N}.$$

PMI is also stated as

Suppose $P(n)$ is a statement for each natural number n. If

$$\begin{cases} P(1) \text{ is true,} \\ \text{Whenever } P(k) \text{ is true, } P(k+1) \text{ is true.} \end{cases}$$

then

$$P(n) \text{ is true for all } n \in \mathbf{N}.$$

PMI is used :

- Proving formulas such as

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Recursive definitions:

$$\begin{cases} 1! = 1 \\ n! = n \cdot (n-1)!, \quad n > 1. \end{cases}$$

There are two statements which are equivalent to PMI.

- *Well Ordering Principle* (WOP): Every nonempty subset A of $\mathbf{N}$ has a *least element* – a number $a \in A$ such that $a \leq x$ for all $x \in A$.

- *Principle of Complete Induction* (PCI):

  If A is a subset of $\mathbf{N}$ such that

  $$\begin{cases} 1 \in A, \\ \text{Whenever } 1, \ldots, k \in A,\ k+1 \in A, \end{cases}$$

  then

  $$A = \mathbf{N}.$$

Integers Z and Rational Numbers Q

The natural numbers $\mathbf{N}$ seem to satisfy

- P1 (addition is associative) Yes
- P2 (zero) No
- P3 (additive inverse) No
- P4 (commutative addition) Yes
- P5 (multiplication is associative) Yes
- P6 (one) Yes
- P7 (multiplicative inverse) No
- P8 (commutative multiplication) Yes
- P9 (distributive) Yes

The *positive set* $P = \mathbf{N}$ satisfies

- P10 (trichotomy) Yes
- P11 ($P + P \subset P$) Yes
- P12 ($P \cdot P \subset P$) Yes

By adjoining 0 and the *negative integers*, with some effort we can recover P2 and P3. For the *integers*, $\mathbf{Z}$, the positive set which satisfies P10, P11, and P12 is still $\mathbf{N}$.

The *rational numbers*, $\mathbf{Q}$, are objects of the form $\frac{p}{q}$, p, q integers, $q \neq 0$. The set P of *positive rational numbers* is ... (left to the reader).

The Rationals are not All Real Numbers

We shall rely on the following fact outlined in Chapter 2, Problem 17:

Every natural number n satisfies one and only one of the following

$$\begin{cases} n = 1, \\ n \text{ is a prime number}, \\ n = ab, \; a, b \in \mathbf{N}, \; 1 < a, b < n. \end{cases}$$

The definition of a prime number is given in Chapter 2, Problem 17.

Definition. *A natural number p is called a prime number if it is impossible to write $p = ab$ for natural numbers a and b unless one of these is p and the other 1. For convenience, we also agree that 1 is not a prime number.*

Theorem. *Every natural number $n > 1$ can be written as a product of primes. The factorization is unique except for the order of the factors (not proved here).*

Thus every (positive) rational number r can be written as

$$r = \frac{p}{q},$$

where p and q are 1 or are are written as products of primes with no common factors.

Theorem. *There is no rational number r such that*

$$r^2 = 2.$$

Proof: If $r^2 = 2$, write $r = \dfrac{p}{q}$, where p and q are 1 or are written as products of primes with no common factors. Then

$$\left(\frac{p}{q}\right)^2 = 2,$$
$$p^2 = 2q^2,$$

so that 2 is a factor of p^2 and p. Thus $p = 2k$ and $2q^2 = 4k^2$ so that 2 is a factor of q^2 and q also.

Remark: The prove of the above theorem is an example of *proof by contradiction*. What is the contradiction? The statement

p and q are 1 or are written as products of primes with no common factors of 2.

is both *true* and *not true*.

A real number which is *not a rational number* is called an *irrational number*.

N. B. There is an argument (Spivak, Chapter 2, Prob. 17) which shows that

Theorem. *Suppose that $M > 1$, $t > 1$, are natural numbers. Then the t^{th} root of M,*

$$\sqrt[t]{M}$$

is irrational unless $M = p^t$ for some natural number p.

Theorem. *Suppose that p, q, t, M are natural numbers such that $M > 1$, p and q have no common factor and*

$$\left(\frac{p}{q}\right)^t = M.$$

Then

$$M = p^t.$$

Proofs by Induction

We wish to prove that a statement [proposition] $P(n)$ is true for all n. $P(n)$ may be a sentence, formula, property, ...

A good proof by PMI requires:

- State precisely and unambiguously $P(n)$.
- Prove the *base case*: $P(1)$ is true. This step usually requires only a close look at the meaning of the statement "$P(1)$ is true."
- The *induction step*: $P(k)$ implies $P(k+1)$.
  - Write down and assume $P(k)$.
  - Perform any *valid operations* to arrive at $P(k+1)$. It may help to say "We are to show that $P(k+1)$ (write it down!) is true." Remember that you are free to use *valid operations* on $P(k)$ and even $P(1), \ldots, P(k)$.