MthT 430 Notes Chapter 4a Graphs

Real Numbers and Points on a Line

The real numbers in **R** are identified with points on a horizontal line. For the time being, we will identify a real number x with a *decimal expansion*.

- Every decimal expansion represents a real number x:

$$x = \pm N.d_1 d_2 \ldots ,$$
$$d_k \in \{0, 1, ..., 9\} .$$

This is the statement that every infinite series of the form

$$d_1 10^{-1} + d_2 10^{-2} + \ldots , \quad d_k \in \{0, 1, ..., 9\} ,$$

converges.

Just as well we could identify a real number x with a *binary expansion*.

- Every binary expansion represents a real number x:

$$x = \pm N ._{\text{bin}} b_1 b_2 \ldots ,$$
$$b_k \in \{0, 1\} .$$

This is the statement that every infinite series of the form

$$b_1 2^{-1} + b_2 2^{-2} + \ldots , \quad b_k \in \{0, 1\} ,$$

converges.

A demonstration of a correspondence between the binary expansion and a point on a horizontal line was given in class. See also

`chap4b.pdf`

Intervals

- $(a, b) \equiv \{x \mid a < x < b\}$ is the *open interval* from a to b. Usually it is assumed that a is less than b. If $b < a$, then $(a, b) = \emptyset$, the empty interval.

- $[a, b] \equiv \{x \mid a \leq x \leq b\}$ is the *closed interval* from a to b. Usually it is assumed that a is less than or equal b. If $b < a$, then $[a, b] = \emptyset$, the empty interval.

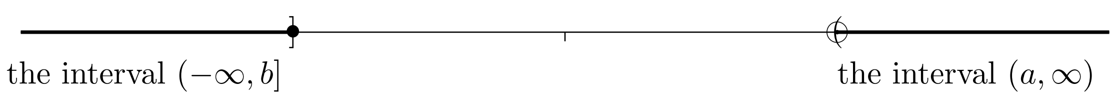
  The "points" $-\infty$ and ∞ are introduced so that we have

- $(a, \infty) \equiv \{x \mid a < x\}$ is the *open interval* from a to ∞.

- $(-\infty, b] \equiv \{x \mid x \leq b\}$ is the *closed interval* from $-\infty$ to b.

- $(-\infty, \infty) \equiv \ldots$

In graphing an interval, whether an endpoint is included or not is usually indicated by explicitly drawing the point or placing a (,), [,], $\bullet$, or $\circ$ at the indicated coordinate.



See the examples in Spivak, pp. 50 ff.

Note the equations (formulas) for lines.

If the function is defined on the closed interval between two points points $x, x + \Delta x$, let

$$\Delta f(x) = f\,(x + \Delta x) - f\,(x)\,.$$

Qualitative properties of graphs to be observed on intervals are

- Continuity (to be defined precisely in Chapter 5) – $\Delta f(x)$ is small if Δx is small. Note particular points where the function is *not* defined or is *not* continuous

- Monotonicity – increasing or decreasing on particular intervals – $\Delta f(x)$ is of constant sign for all $x, x + \Delta x$ in the interval with $\Delta x > 0$

- Concavity on particular intervals – for equal Δx, $\Delta f(x)$ is increasing/decreasing as x increases in the interval.

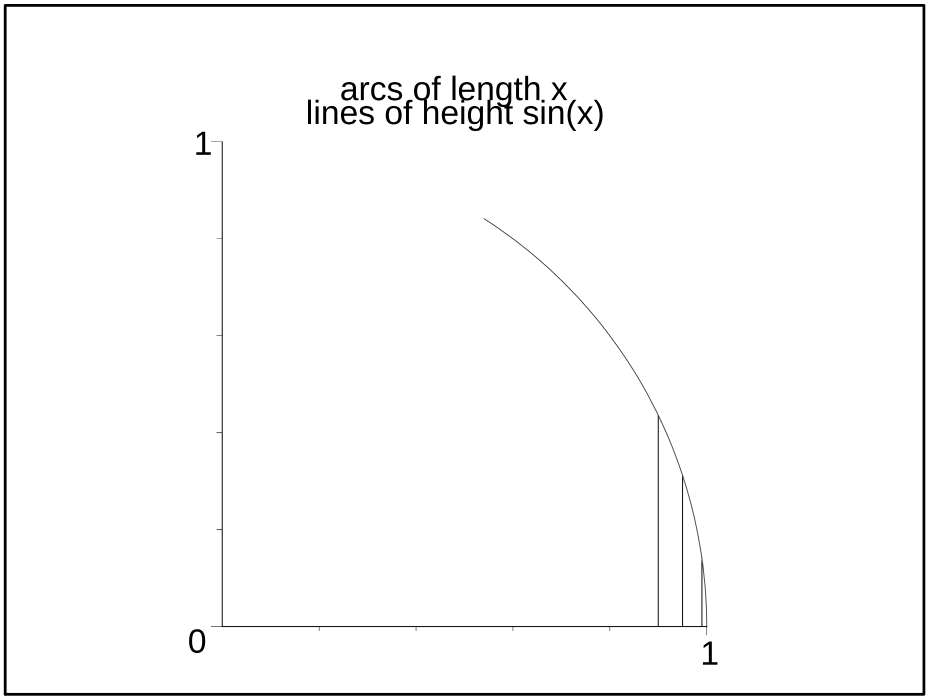
Examples

- Figure 20:
$$f(x) = \sin\left(\frac{1}{x}\right).$$
With our convention, domain $(f) = \{x \neq 0\}$. The intervals of increase and decrease are evident, concavity is not quite so clear.

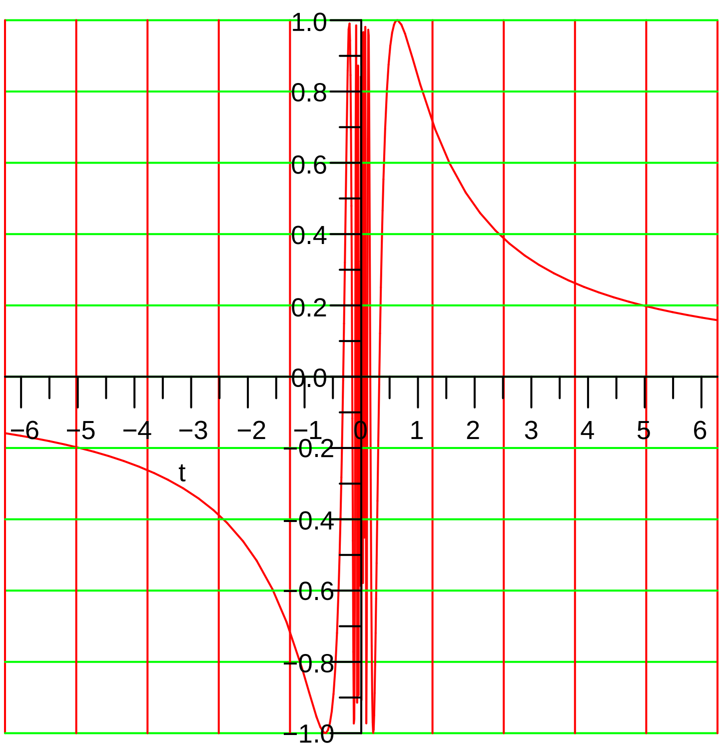
- The function sinxoverx:
$$\text{sinxoverx}\,(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ \text{undefined}, & x = 0. \end{cases}$$

- The function Siprime:
$$\text{Siprime}(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$
The function Siprime(x) is an extension of the function sinxoverx, and is continuous at $x = 0$. At $x = 0$, for Δx small about $\neq 0$,
$$\begin{aligned} \Delta\text{Siprime} &= \text{Siprime}(\Delta x) - \text{Siprime}(0) \\ &= \frac{\sin(\Delta x)}{\Delta x} - 1 \\ &= \frac{|\text{small line}|}{|\text{small arc}|} - 1 \\ &= \text{small – draw a picture.} \end{aligned}$$

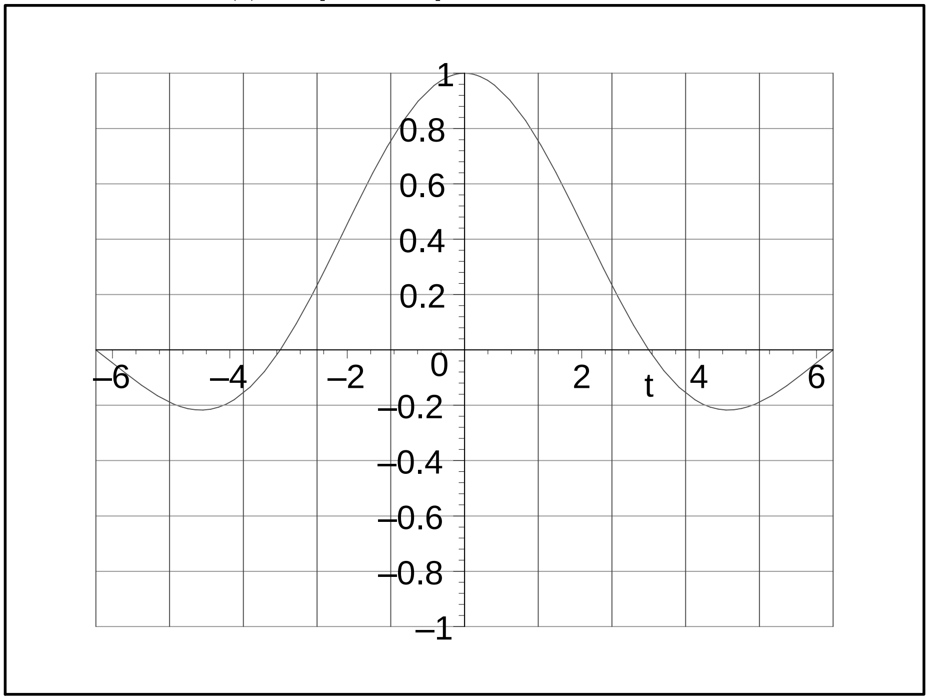
Here is the *picture* for the above calculation:



Here is the plot of $\sin\left(\frac{1}{x}\right)$ on $[-2\pi, 2\pi]$

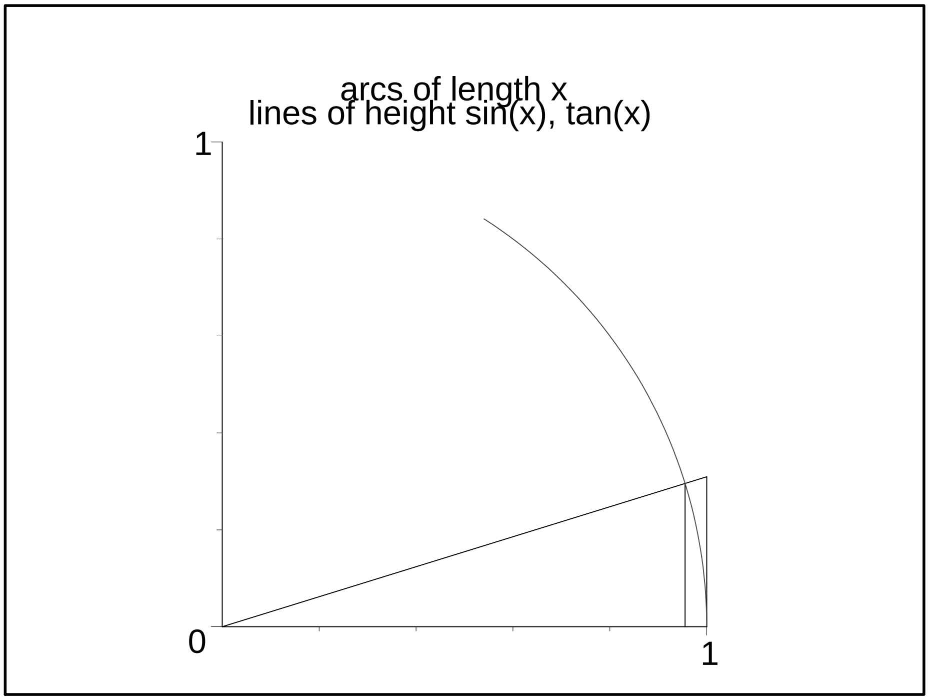


Here is the plot of Siprime(t) on $[-2\pi, 2\pi]$



Here is another picture which shows that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$



Note that, for $x > 0$, $\sin(x) < x < \tan(x)$.