MthT 430 Notes Chapter 5a Limits

Notation

The expression

$$\lim_{x \to a} f(x) = L$$

is read

- The limit of f at $x = a$ is L.
- The limit as x approaches a of $f(x)$ is L.
- The limit of $f(x)$ is L as x approaches a.
- $f(x)$ approaches L as x approaches a.
- The function f approaches the limit L near a (Note: no mention of x).
- (Briefer – p. 99) f approaches L near a.

Meaning

The meaning of the phrase is

Provisional Definition. *(p. 90) The function f approaches the limit L near a, if we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close to (but $\neq$) a.*

- (Somewhat Informal) The function f approaches the limit L near a, if $f(x) - L$ *is small* whenever $x - a$ *is small enough (but $x \neq a$).*
- (Different Words – Somewhat Informal) The function f approaches the limit L near a, if $f(x) = L +$ *small* whenever $x = a +$ *small enough* (but $x \neq a$).
- (Informal) The function f approaches the limit L near a, if $f(x)$ is *close to* L whenever x is *close enough to* (but $\neq$) a.
- (Explanation of Provisional) You tell me how close you want $f(x)$ to be to L and I will tell you how close x needs to be to a to force $f(x)$ to be as close to L as you requested.
- (Explanation of Different Words – Somewhat Informal) $f(x) = L +$ small means that size of $f(x) - L$ is *small* in the sense that, $f(x) - L$ is as small as we like (whether $.1, .00001, 10^{-100}, \ldots$), by imposing that $|x - a|$ is *small enough* (but $\neq 0$). How *small* is *small enough* for $x - a$ depends on *how small* we require $f(x) - L$ to be.
- (More Explanation of Provisional JL) Given a positive size [number] ϵ, there is a positive

size [number] δ such that if the size of $x - a$ is less than δ (but not 0, then the size of $f(x) - L$ is less than ϵ. Here the *size* of a number is its absolute value.

Definition of Limit

Definition. *(p. 96) The function f approaches the limit L near a means: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*

Different Words. *(p. 96) The function f approaches the limit L near a means: For every desired degree of closeness $\epsilon > 0$, there is a degree of closeness $\delta > 0$ such that, for all $x \neq a$, if $x - a$ is within δ of a, then $f(x)$ is within ϵ of L.*

The phrase α is within ϵ of β means: $|\alpha - \beta| < \epsilon$.

Change of Notation. *The function f approaches the limit L near a means: For every $\clubsuit > 0$, there is some $\heartsuit > 0$ such that, for all $\spadesuit$, if $0 < |\spadesuit - a| < \heartsuit$, then $|f(\spadesuit) - L| < \clubsuit$.*

Fundamental Properties of Limits

Theorem 1. The limit is unique. *If f approaches L near a, and f approaches M near a, then $L = M$.*

Informal Proof: For x near enough to a, $f(x)$ is very close to both L and M. By the triangle inequality,

$$\begin{aligned} |L - M| &= |(L - f(x)) + (f(x) - M)| \\ &\leq |L - f(x)| + |f(x) - M| \\ &= \text{small} + \text{small} \\ &= \text{small}. \end{aligned}$$

Thus for $x - a$ small enough, $|L - M|$ is as small as desired. Conclude $L = M$.

Fact. *A number $Y = 0$ iff for very $\epsilon > 0$, $|Y| < \epsilon$.*

Proof: (Text, p. 98.)

Theorem 2. *If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then*

$$\lim_{x \to a} (f + g)(x) = L + M,$$
$$\lim_{x \to a} (f \cdot g)(x) = L \cdot M.$$

If $M \neq 0$, then

$$\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{M}.$$

Proof. See Spivak, Problems 1.20 ff.

Discussion before the proof: Let's do the result for products. We can make (how? – by requiring $x-a$ to be small enough (and $\neq 0$) $f(x) = L + \text{small}_f$ and $g(x) = M + \text{small}_g$. Then for $x = a + $ small enough, $x \neq a$,

$$\begin{aligned} f(x) \cdot g(x) &= (L + \text{small}_f) \cdot (M + \text{small}_g) \\ &= L \cdot M + \text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g \\ &= L \cdot M + \text{Remainder}. \end{aligned}$$

Now it is evident that Remainder can be made as small as we like by requiring $|x-a|$ sufficiently small (but $\neq 0$).

The Proof: Given $\epsilon > 0$, we have

$$|f(x) \cdot g(x) - L \cdot M| = |\text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g|,$$

where $\text{small}_f = f(x) - L$, $\text{small}_g = g(x) - M$. Now choose $\delta > 0$ so that whenever $0 < |x-a| < \delta$,

$$\begin{aligned} |\text{small}_f| &= |f(x) - L| < \epsilon, \\ |\text{small}_g| &= |g(x) - M| < \epsilon. \end{aligned}$$

Then whenever $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) \cdot g(x) - L \cdot M| &= |\text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g|, \\ &\leq |\epsilon \cdot M| + |\epsilon \cdot L| + \epsilon^2. \end{aligned} \tag{$*$}$$

Now if we also assume that $\epsilon < 1$, we have that

$$(*) \leq \epsilon \cdot (|M| + |L| + 1),$$

and it is evident that $|f(x) - L|$ can be made as small as desired. There are a couple of ways:

- Choose the δ that works for $\hat{\epsilon} = \dfrac{\epsilon}{(|M| + |L| + 1)} > 0$.
- Use a modified equivalent definition of limit: The function f approaches the limit L near a means: There is an $\epsilon_0 > 0$ and a $K > 0$ such that : For every ϵ, $\epsilon_0 > \epsilon > 0$, there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - L| < K \cdot \epsilon$.

Notes

- Given $\epsilon > 0$, the δ such that $0 < |x - a| < \delta$ assures $|f(x) - L| < \epsilon$ usually depends on ϵ, as well as depending on the point a and function f and all of its properties. Finding an explicit expression for the *optimal* δ is not required nor necessarily interesting unless doing numerical error estimates.

- In the product and quotient example, the $\delta = \delta_\epsilon$ was chosen with the additional requirement that $\epsilon < 1$.

- Pay attention to the domain of the function. See the *technical detail* on p. 102.

- Observe the definitions of *one sided limits* – also called *limits from above [below]* and *limits from the left [right]*.

Thinking About Limits

Definition. *(Actual, p. 96)*

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Definition - Working JL II.

$$\lim_{x \to a} f(x) = L$$

means:

- *For all* $x = a + \text{smallenufneq0}$, x *is in* $\text{domain} f$.

- $f(x) = L + \text{assmallasdesired}$, *for* $x = a + \text{smallenufneq0}$.

  Translation:

- assmallasdesired means, given $\epsilon > 0$, then $|\text{assmallasdesired}| < \epsilon$ is the desired result.

- smallenufneq0 means, find a $\delta > 0$ such that $0 < |\text{smallenufneq0}| < \delta$ is the sufficient condition.

- smallenuf means, find a $\delta > 0$ such that $|\text{smallenufneq0}| < \delta$ is the sufficient condition.

Definition - Working JL II'.

$$\lim_{x \to a} f(x) = L$$

means:

- *For* $|x - a|$ smallenufneq0, x *is in* $\text{domain} f$.

- $f(x) - L$ *is* assmallasdesired, *for* $x - a$ *is* smallenufneq0.