MthT 430 Notes Chapter 5a Limits

Notation

The expression
\[ \lim_{x \to a} f(x) = L \]
is read

• The limit of \( f \) at \( x = a \) is \( L \).
• The limit as \( x \) approaches \( a \) of \( f(x) \) is \( L \).
• The limit of \( f(x) \) is \( L \) as \( x \) approaches \( a \).
• \( f(x) \) approaches \( L \) as \( x \) approaches \( a \).
• The function \( f \) approaches the limit \( L \) near \( a \) (Note: no mention of \( x \)).
• (Briefer – p. 99) \( f \) approaches \( L \) near \( a \).

Meaning

The meaning of the phrase is

**Provisional Definition.** (p. 90) The function \( f \) approaches the limit \( L \) near \( a \), if we can make \( f(x) \) as close as we like to \( L \) by requiring that \( x \) be sufficiently close to (but \( \neq \)) \( a \).

• (Somewhat Informal) The function \( f \) approaches the limit \( L \) near \( a \), if \( f(x) - L \) is small whenever \( x - a \) is small enough (but \( x \neq a \)).
• (Different Words – Somewhat Informal) The function \( f \) approaches the limit \( L \) near \( a \), if \( f(x) = L + \) small whenever \( x = a + \) small enough (but \( x \neq a \)).
• (Informal) The function \( f \) approaches the limit \( L \) near \( a \), if \( f(x) \) is close to \( L \) whenever \( x \) is close enough to (but \( \neq \)) \( a \).
• (Explanation of Provisional) You tell me how close you want \( f(x) \) to be to \( L \) and I will tell you how close \( x \) needs to be to \( a \) to force \( f(x) \) to be as close to \( L \) as you requested.
• (Explanation of Different Words – Somewhat Informal) \( f(x) = L + \) small means that size of \( f(x) - L \) is small in the sense that, \( f(x) - L \) is as small as we like (whether \( .1, .00001, 10^{-100}, \ldots \)), by imposing that \( |x - a| \) is small enough (but \( \neq 0 \)). How small is small enough for \( x - a \) depends on how small we require \( f(x) - L \) to be.
• (More Explanation of Provisional JL) Given a positive size [number] \( \epsilon \), there is a positive
size \([\text{number}]\) \(\delta\) such that if the size of \(x - a\) is less than \(\delta\) (but not 0, then the size of \(f(x) - L\) is less than \(\epsilon\). Here the size of a number is its absolute value.

### Definition of Limit

**Definition.** (p. 96) The function \(f\) approaches the limit \(L\) near \(a\) means: For every \(\epsilon > 0\), there is some \(\delta > 0\) such that, for all \(x\), if \(0 < |x - a| < \delta\), then \(|f(x) - L| < \epsilon\).

**Different Words.** (p. 96) The function \(f\) approaches the limit \(L\) near \(a\) means: For every desired degree of closeness \(\epsilon > 0\), there is a degree of closeness \(\delta > 0\) such that, for all \(x \neq a\), if \(x - a\) is within \(\delta\) of \(a\), then \(f(x)\) is within \(\epsilon\) of \(L\).

The phrase \(\alpha\) is within \(\epsilon\) of \(\beta\) means: \(|\alpha - \beta| < \epsilon\).

**Change of Notation.** The function \(f\) approaches the limit \(L\) near \(a\) means: For every \(♣ > 0\), there is some \(♥ > 0\) such that, for all \(♠\), if \(0 < |♠ - a| < ♥\), then \(|f(♠) - L| < ♣\).

### Fundamental Properties of Limits

**Theorem 1.** The limit is unique. If \(f\) approaches \(L\) near \(a\), and \(f\) approaches \(M\) near \(a\), then \(L = M\).

**Informal Proof:** For \(x\) near enough to \(a\), \(f(x)\) is very close to both \(L\) and \(M\). By the triangle inequality,

\[
|L - M| = |(L - f(x)) + (f(x) - M)|
\leq |L - f(x)| + |f(x) - M|
= \text{small} + \text{small}
= \text{small}.
\]

Thus for \(x - a\) small enough, \(|L - M|\) is as small as desired. Conclude \(L = M\).

**Fact.** A number \(Y = 0\) iff for very \(\epsilon > 0\), \(|Y| < \epsilon\).

**Proof:** (Text, p. 98.)

**Theorem 2.** If \(\lim_{x \to a} f(x) = L\) and \(\lim_{x \to a} g(x) = M\), then

\[
\lim_{x \to a} (f + g)(x) = L + M,
\]

\[
\lim_{x \to a} (f \cdot g)(x) = L \cdot M.
\]

If \(M \neq 0\), then

\[
\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{M}.
\]
Proof. See Spivak, Problems 1.20 ff.

Discussion before the proof: Let’s do the result for products. We can make (how? – by requiring $x - a$ to be small enough (and $\neq 0$) $f(x) = L + \text{small}_f$ and $g(x) = M + \text{small}_g$. Then for $x = a + \text{small enough}$, $x \neq a$,

$$f(x) \cdot g(x) = (L + \text{small}_f) \cdot (M + \text{small}_g)$$

$$= L \cdot M + \text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g$$

$$= L \cdot M + \text{Remainder}.$$  

Now it is evident that Remainder can be made as small as we like by requiring $|x - a|$ sufficiently small (but $\neq 0$).

The Proof: Given $\epsilon > 0$, we have

$$|f(x) \cdot g(x) - L \cdot M| = |\text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g|,$$

where $\text{small}_f = f(x) - L$, $\text{small}_g = g(x) - M$. Now choose $\delta > 0$ so that whenever $0 < |x - a| < \delta$,

$$|\text{small}_f| = |f(x) - L| < \epsilon,$$

$$|\text{small}_g| = |g(x) - M| < \epsilon.$$

Then whenever $0 < |x - a| < \delta$,

$$|f(x) \cdot g(x) - L \cdot M| = |\text{small}_f \cdot M + L \cdot \text{small}_g + \text{small}_f \cdot \text{small}_g|,$$

$$\leq |\epsilon \cdot M| + |\epsilon \cdot L| + \epsilon^2.$$  \hspace{1cm} (\ast)  

Now if we also assume that $\epsilon < 1$, we have that

$$\ast \leq \epsilon \cdot (|M| + |L| + 1),$$

and it is evident that $|f(x) - L|$ can be made as small as desired. There are a couple of ways:

- Choose the $\delta$ that works for $\hat{\epsilon} = \frac{\epsilon}{(|M| + |L| + 1)} > 0$.

- Use a modified equivalent definition of limit: The function $f$ approaches the limit $L$ near $a$ means: There is an $\epsilon_0 > 0$ and a $K > 0$ such that: For every $\epsilon$, $\epsilon_0 > \epsilon > 0$, there is a $\delta > 0$ such that, for all $x$, if $0 < |x - a| < \delta$, then $|f(x) - L| < K \cdot \epsilon$.  

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Notes

• Given $\epsilon > 0$, the $\delta$ such that $0 < |x - a| < \delta$ assures $|f(x) - L| < \epsilon$ usually depends on $\epsilon$, as well as depending on the point $a$ and function $f$ and all of its properties. Finding an explicit expression for the optimal $\delta$ is not required nor necessarily interesting unless doing numerical error estimates.

• In the product and quotient example, the $\delta = \delta_\epsilon$ was chosen with the additional requirement that $\epsilon < 1$.

• Pay attention to the domain of the function. See the technical detail on p. 102.

• Observe the definitions of one sided limits – also called limits from above /below/ and limits from the left /right/.

Thinking About Limits

Definition. (Actual, p. 96)

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all $x$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Definition - Working JL II.

$$\lim_{x \to a} f(x) = L$$

means:

• For all $x = a$ + smallenufneq0, $x$ is in domain $f$.

• $f(x) = L$ + assmallasdesired, for $x = a$ + smallenufneq0.

Translation:

• assmallasdesired means, given $\epsilon > 0$, then $|assmallasdesired| < \epsilon$ is the desired result.

• smallenufneq0 means, find a $\delta > 0$ such that $0 < |smallenufneq0| < \delta$ is the sufficient condition.

• smallenuf means, find a $\delta > 0$ such that $|smallenufneq0| < \delta$ is the sufficient condition.

Definition - Working JL II’.

$$\lim_{x \to a} f(x) = L$$

means:

• For $|x - a|$ smallenufneq0, $x$ is in domain $f$.

• $f(x) - L$ is assmallasdesired, for $x - a$ is smallenufneq0.