Binary Expansion Arguments

Consider the following problem:

1. Let \( f(x) \) be a function such that
   - domain \( f = [0, 1) \).
   - For all \( x \) (in \( [0, 1) \)), \( 0 \leq f(x) < 1 \).
   - The function \( f \) is increasing on \( [0, 1) \).

Show that there is a number \( L \), \( 0 \leq L \leq 1 \), such that

\[
\lim_{x \to 1^-} f(x) = L.
\]

**Hint:** Construct a binary expansion for \( L \).

A picture is helpful!

To find the expansion for \( L \), ask the question:

Is there an \( x \in [0, 1) \) such that \( f(x) \geq \frac{1}{2} = 0.\text{bin}1 \)?
If NO, let $x_1 = 0$, $b_1 = 0$, $s_1 = 0$. If YES, let $x_1 = x$, $b_1 = 1$, $s_1 = 0.1$. In both cases, for $x_1 \leq x < 1$, $s_1 \leq f(x_1) \leq f(x) \leq s_1 + \frac{1}{2}$.

Next divide the interval $[s_1, s_1 + \frac{1}{2}]$ into two parts $[0, s_1 + 0.1]$ and $[0.1, s_1 + \frac{1}{2}]$. 
Ask the question:

Is there an \( x \in [x_1, 1) \) such that \( f(x) \geq s_1 + \frac{1}{2^2} = 0_{\text{bin}} b_11 \)\(^1\)?

If NO, let \( x_2 = x_1, b_2 = 0, s_2 = 0_{\text{bin}} b_1 b_2 \)\(^2\). If YES, let \( x_2 = x, b_1 = 1, s_2 = 0_{\text{bin}} b_1 b_2 = s_1 + \frac{1}{2^2} \). Then for \( x_2 \leq x < 1, s_2 \leq f(x_2) \leq f(x) \leq s_2 + \frac{1}{2^2} \).

\(^1\) Thinking about this later, I noticed that \( s_1 + \frac{1}{2^2} \) is the *midpoint* of the new interval under consideration.

\(^2\) In this case \( x_2 = x_1 \) and \( s_2 = s_1 \).
If $x_1, \ldots, x_n, b_1, \ldots b_n, s_n = 0.b_1 \ldots b_n$ have been constructed so that for $x_n \leq x < 1$, $s_n \leq f(x_n) \leq f(x) \leq s_n + \frac{1}{2^n}$.

Ask the question: Is there an $x \in [x_n, 1)$ such that $f(x_{n+1}) \geq s_n + \frac{1}{2^{n+1}} = 0.b_1 \ldots b_n1$?

Then let

$$x_{n+1} = \begin{cases} x_n, & \text{NO}, \\ x, & \text{YES}, \end{cases}$$

$$b_{n+1} = \begin{cases} 0, & \text{NO}, \\ 1, & \text{YES}, \end{cases}$$

$$s_{n+1} = 0.b_1b_2 \ldots b_{n+1}$$

$$= b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_{n+1} \cdot 2^{-(n+1)}$$

For $x_{n+1} \leq x < 1$, $s_{n+1} \leq f(x_{n+1}) \leq f(x) \leq s_{n+1} + \frac{1}{2^{n+1}}$.

Then

$$L = 0.b_1b_2 \ldots b_n \ldots = \lim_{n \to \infty} s_n$$

since for all $x$, if $x_n \leq x < 1$,

$$s_n \leq f(x) \leq L \leq s_n + \frac{1}{2^n}$$

and

$$0 \leq L - f(x) < \frac{1}{2^n}.$$