Sequences
Cf. Spivak Chapter 22.

Definition. An infinite sequence is a function whose domain is \( \mathbb{N} \).

As a convention, we also allow the domain of a sequence to be a subset of \( \mathbb{N} \) which includes all natural numbers sufficiently large.

Notation

If \( a \) is the name of the sequence, instead of listing the particular values by

\[ a(1), a(2), \ldots, \]

we almost always use the subscript notation

\[ a_1, a_2, \ldots. \]

We denote the sequence by

\[ \{a_n\} \]

Limits of sequences

Definition. A sequence \( \{a_n\} \) converges to \( L \) (in symbols \( \lim_{n \to \infty} a_n = L \)) iff for every \( \epsilon > 0 \), there is a natural number \( N \) such that, for all natural numbers \( n \),

\[ \text{if } n > N, \text{ then } |a_n - L| < \epsilon. \]

A sequence \( \{a_n\} \) is said to converge if it converges to \( L \) for some [finite!] number \( L \), and to diverge if it does not converge.

Compare

- For a function \( f \) whose domain includes all \( x \) sufficiently large and positive,

\[ \lim_{x \to \infty} f(x) = L. \]

- For a sequence \( \{a_n\} \), whose domain includes all \( n \) sufficiently large and positive,

\[ \lim_{n \to \infty} a_n = L. \]