This note is an attempt to clear up the confusion I (JL) probably created at the beginning of class October 10, 2007.

It is also closely related to problems 5.25 and 5.26 in Spivak.

In applying the definition of \( \lim_{x\to a} f(x) = L \), it is sometimes convenient to multiply \( \epsilon \) and/or \( \delta \) by positive constants.

**Limits – Definitions**

**Definition (Actual, p. 96).**

\[
\lim_{x \to a} f(x) = L
\]

**means:** For every \( \epsilon > 0 \), there is some \( \delta > 0 \) such that, for all \( x \), if \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \epsilon \).

**An Equivalent Definition of Limit**

**Definition 10^6.**

\[
\lim_{x \to a} f(x) = L
\]

**means:** For every \( \epsilon > 0 \), there is some \( \delta > 0 \) such that, for all \( x \), if \( 0 < |x - a| < 10^6 \delta \), then \( |f(x) - L| < \epsilon \).

10^6 \Rightarrow **Actual:** Fix \( \epsilon > 0 \). By 10^6, there is a \( \delta > 0 \) such that, for all \( x \), if \( 0 < |x - a| < 10^6 \delta \), then \( |f(x) - L| < \epsilon \). If \( 0 < |x - a| < \delta \), then \( 0 < |x - a| < 10^6 \delta \), so \( |f(x) - L| < \epsilon \).

**Actual \Rightarrow 10^6:** Fix \( \epsilon > 0 \). By **Actual**, there is an \( \eta > 0 \) such that, for all \( x \), if \( 0 < |x - a| < \eta \), then \( |f(x) - L| < \epsilon \). Let \( \delta = 10^{-6} \eta \). Then \( \delta > 0 \) iff \( \eta > 0 \). For all \( x \), if \( 0 < |x - a| < 10^6 \delta = \eta \), \( |f(x) - L| < \epsilon \).

10^6 \Rightarrow **Actual:** (another Proof) Fix \( \epsilon > 0 \). By 10^6, there is a \( \rho > 0 \) such that, for all \( x \), if \( 0 < |x - a| < 10^6 \rho \), then \( |f(x) - L| < \epsilon \). Let \( \delta = 10^6 \rho \). Then \( \delta > 0 \) iff \( \rho > 0 \). For all \( x \), if \( 0 < |x - a| < \delta = 10^6 \rho \), \( |f(x) - L| < \epsilon \).