MthT 430 Chapter 5 Limits – An Equivalent Definition

This note is an attempt to clear up the confusion I (JL) probably created at the beginning of class October 10, 2007.

It is also closely related to problems 5.25 and 5.26 in Spivak.

In applying the definition of $\lim_{x\to a} f(x) = L$, it is sometimes convenient to multiply ϵ and/or δ by positive constants.

Limits – Definitions

Definition (Actual, p. 96).

$$\lim_{x\to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

An Equivalent Definition of Limit

Definition 10^6.

$$\lim_{x\to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < 10^6\delta$, *then* $|f(x) - L| < \epsilon$.

$10^6 \Rightarrow$ **Actual**: Fix $\epsilon > 0$. By 10^6, there is a $\delta > 0$ such that, for all x, if $0 < |x - a| < 10^6\delta$, then $|f(x) - L| < \epsilon$. If $0 < |x - a| < \delta$, then $0 < |x - a| < 10^6\delta$, so $|f(x) - L| < \epsilon$.

Actual $\Rightarrow 10^6$**:** Fix $\epsilon > 0$. By **Actual**, there is an $\eta > 0$ such that, for all x, if $0 < |x - a| < \eta$, then $|f(x) - L| < \epsilon$. Let $\delta = 10^{-6}\eta$. Then $\delta > 0$ iff $\eta > 0$. For all x, if $0 < |x - a| < 10^6\delta = \eta$, $|f(x) - L| < \epsilon$.

$10^6 \Rightarrow$ **Actual:** (another Proof) Fix $\epsilon > 0$. By 10^6, there is a $\rho > 0$ such that, for all x, if $0 < |x - a| < 10^6\rho$, then $|f(x) - L| < \epsilon$. Let $\delta = 10^6\rho$. Then $\delta > 0$ iff $\rho > 0$. For all x, if $0 < |x - a| < \delta = 10^6\rho$, $|f(x) - L| < \epsilon$.