MthT 430 Notes Chapter 6a Binary Expansions and Arguments

Real Numbers and Binary Expansions

The real numbers in $\mathbf{R}$ are identified with points on a horizontal line. For the time being, we will identify a real number x with a *decimal expansion*.

- Every decimal expansion represents a real number x:

$$x = \pm N.d_1 d_2 \ldots ,$$
$$d_k \in \{0, 1, ..., 9\} .$$

This is the statement that every infinite series of the form

$$d_1 10^{-1} + d_2 10^{-2} + \ldots , \quad d_k \in \{0, 1, ..., 9\} ,$$

converges.

Just as well we could identify a real number x with a *binary expansion*.

- Every binary expansion represents a real number x:

$$x = \pm N ._{\text{bin}} b_1 b_2 \ldots ,$$
$$b_k \in \{0, 1\} .$$

This is the statement that every infinite series of the form

$$b_1 2^{-1} + b_2 2^{-2} + \ldots , \quad b_k \in \{0, 1\} ,$$

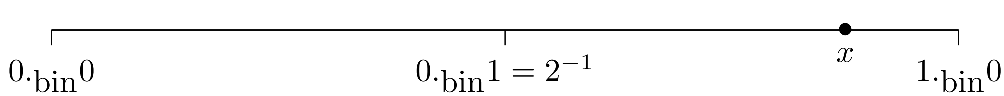
converges.

A demonstration of a correspondence between the binary expansion and a point on a horizontal line was given in class.

Constructing the Binary Expansion

Let x be a real number, $0 \le x < 1$.

We divide $[0, 1)$ into two *half–open* intervals, $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$.



Notice, that in binary notation we may write the two intervals respectively as $\big[0, 0._{\text{bin}}1\big)$ and $\big[0._{\text{bin}}1, 1\big)$.

If x is in the left interval, $x = 0 + 0 \cdot 2^{-1} + \ldots$, so we let $b_1 = 0$, $s_1 = 0._{\text{bin}}b_1$, so that

$$\begin{aligned} x &= 0._{\text{bin}}0 + \ldots \\ &= s_1 + r_1, \end{aligned}$$

where

$$0 \le r_1 < 2^{-1}.$$

If x is in the right interval, $x = 0 + 1 \cdot 2^{-1} + \ldots$, so we let $b_1 = 1$, $s_1 = 0._{\text{bin}}b_1$, so that

$$\begin{aligned} x &= 0._{\text{bin}}1 + \ldots \\ &= s_1 + r_1, \end{aligned}$$

where

$$0 \le r_1 < 2^{-1}.$$

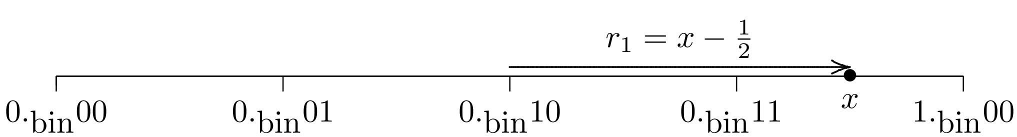
We formalize the process by saying

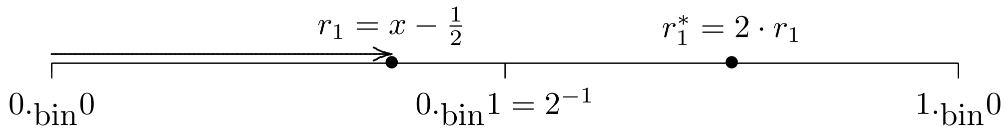
$$\begin{aligned} b_1 &= \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x < 1, \end{cases} \\ s_1 &= 0._{\text{bin}}b_1 \\ &= b_1 \cdot 2^{-1}, \\ x &= s_1 + r_1, \\ 0 &\le r_1 < 2^{-1}. \end{aligned}$$

If the first *remainder* r_1 is 0, that is $x = 0$ or $x = \frac{1}{2}$, STOP; $x = s_1$ and the binary expansion of x has been found.

If $r_1 \neq 0$, we apply a similar process to $r_1 = x - s_1$ to find the second binary digit in the expansion of x. Let $r_1^* = 2^1 \cdot r_1$. Once again divide $[0, 1)$ into the two *half–open* intervals, $\left[0, \dfrac{1}{2}\right)$ and $\left[\dfrac{1}{2}, 1\right)$.

If r_1^* is in the left interval, $x = 0 + b_1 \cdot 2^{-1} + 0 \cdot 2^{-2} + \ldots$; If r_1^* is in the right interval, $x = 0 + b_1 \cdot 2^{-1} + 1 \cdot 2^{-2} + \ldots$, so we let $b_1 = 1$. This is same is saying that x is in the left or right half as the interval selected in step 1.





Thus

$$
\begin{aligned}
b_2 &= \begin{cases} 0, & 0 \le r_1^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \le r_1^* < 1, \end{cases} \\
s_2 &= 0._{\text{bin}} b_1 b_2 \\
&= b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} \\
x &= s_2 + r_2, \\
0 &\le r_2 < 2^{-2}.
\end{aligned}
$$

If the second remainder r_2 is 0, STOP; $x = s_2$ and the binary expansion of x has been found. Otherwise continue!

The continuation may be defined by the Principle of Mathematical Induction (Recursion) so that if b_n, $s_n = 0._{\text{bin}} b_1 \ldots b_n$, $x = s_n + r_n$, have been constructed so that $0 \le r_n < 2^{-n}$, let

$$
\begin{aligned}
r_n^* &= 2^n \cdot r_n, \\
b_{n+1} &= \begin{cases} 0, & 0 \le r_n^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \le r_n^* < 1, \end{cases} \\
s_{n+1} &= 0._{\text{bin}} b_1 b_2 \ldots b_{n+1} \\
&= b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_{n+1} \cdot 2^{-(n+1)} \\
x &= s_{n+1} + r_{n+1}, \\
0 &\le r_{n+1} < 2^{-(n+1)}.
\end{aligned}
$$

If the remainder r_{n+1} is 0, STOP; $x = s_{n+1}$ and the binary expansion of x has been found. Otherwise continue!

What has been done: Given x, $0 \le x < 1$, there is a nondecreasing sequence $\{s_n\}_{n=1}^{\infty}$ of finite binary expansions such that $0 \le x - s_n < 2^{-n}$. Thus

$$\lim_{n \to \infty} s_n = x.$$