Real Numbers and Binary Expansions

The real numbers in $\mathbb{R}$ are identified with points on a horizontal line. For the time being, we will identify a real number $x$ with a decimal expansion.

- Every decimal expansion represents a real number $x$:
  \[ x = \pm N.d_1d_2\ldots, \]
  \[ d_k \in \{0, 1, \ldots, 9\}. \]

This is the statement that every infinite series of the form
\[ d_110^{-1} + d_210^{-2} + \ldots, \quad d_k \in \{0, 1, \ldots, 9\}, \]
converges.

Just as well we could identify a real number $x$ with a binary expansion.

- Every binary expansion represents a real number $x$:
  \[ x = \pm N_{\text{bin}}b_1b_2\ldots, \]
  \[ b_k \in \{0, 1\}. \]

This is the statement that every infinite series of the form
\[ b_12^{-1} + b_22^{-2} + \ldots, \quad b_k \in \{0, 1\}, \]
converges.

A demonstration of a correspondence between the binary expansion and a point on a horizontal line was given in class.

Constructing the Binary Expansion

Let $x$ be a real number, $0 \leq x < 1$.

We divide $[0, 1)$ into two half-open intervals, $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$.
Notice, that in binary notation we may write the two intervals respectively as $[0, 0_{\text{bin}}1)$ and $[0_{\text{bin}}1, 1)$. 

If $x$ is in the left interval, $x = 0 + 0 \cdot 2^{-1} + \ldots$, so we let $b_1 = 0$, $s_1 = 0_{\text{bin}}b_1$, so that

$$x = 0_{\text{bin}}0 + \ldots = s_1 + r_1,$$

where

$$0 \leq r_1 < 2^{-1}.$$

If $x$ is in the right interval, $x = 0 + 1 \cdot 2^{-1} + \ldots$, so we let $b_1 = 1$, $s_1 = 0_{\text{bin}}b_1$, so that

$$x = 0_{\text{bin}}1 + \ldots = s_1 + r_1,$$

where

$$0 \leq r_1 < 2^{-1}.$$

We formalize the process by saying

$$b_1 = \begin{cases} 
0, & 0 \leq x < \frac{1}{2}, \\
1, & \frac{1}{2} \leq x < 1,
\end{cases}$$

$$s_1 = 0_{\text{bin}}b_1 = b_1 \cdot 2^{-1},$$

$$x = s_1 + r_1,$$

$$0 \leq r_1 < 2^{-1}.$$ 

If the first remainder $r_1$ is 0, that is $x = 0$ or $x = \frac{1}{2}$, STOP; $x = s_1$ and the binary expansion of $x$ has been found.

If $r_1 \neq 0$, we apply a similar process to $r_1 = x - s_1$ to find the second binary digit in the expansion of $x$. Let $r^*_1 = 2^1 \cdot r_1$. Once again divide $[0, 1)$ into the two half-open intervals, $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$.

If $r^*_1$ is in the left interval, $x = 0 + b_1 \cdot 2^{-1} + 0 \cdot 2^{-2} + \ldots$; If $r^*_1$ is in the right interval, $x = 0 + b_1 \cdot 2^{-1} + 1 \cdot 2^{-2} + \ldots$, so we let $b_1 = 1$. This is same as saying that $x$ is in the left or right half as the interval selected in step 1.
Thus

\[ b_2 = \begin{cases} 0, & 0 \leq r_1^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r_1^* < 1, \end{cases} \]

\[ s_2 = 0.\text{bin}b_1b_2 \]
\[ = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} \]
\[ x = s_2 + r_2, \]
\[ 0 \leq r_2 < 2^{-2}. \]

If the second remainder \( r_2 \) is 0, STOP; \( x = s_2 \) and the binary expansion of \( x \) has been found. Otherwise continue!

The continuation may be defined by the Principle of Mathematical Induction (Recursion) so that if \( b_n, s_n = 0.\text{bin}b_1 \ldots b_n, x = s_n + r_n, \) have been constructed so that \( 0 \leq r_n < 2^{-n}, \) let

\[ r_n^* = 2^n \cdot r_n, \]
\[ b_{n+1} = \begin{cases} 0, & 0 \leq r_n^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r_n^* < 1, \end{cases} \]
\[ s_{n+1} = 0.\text{bin}b_1b_2 \ldots b_n \]
\[ = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_n \cdot 2^{-(n+1)} \]
\[ x = s_{n+1} + r_{n+1}, \]
\[ 0 \leq r_{n+1} < 2^{-(n+1)}. \]

If the remainder \( r_{n+1} \) is 0, STOP; \( x = s_{n+1} \) and the binary expansion of \( x \) has been found. Otherwise continue!

**What has been done**: Given \( x, 0 \leq x < 1, \) there is a nondecreasing sequence \( \{s_n\}_{n=1}^{\infty} \) of finite binary expansions such that \( 0 \leq x - s_n < 2^{-n}. \) Thus

\[ \lim_{n \to \infty} s_n = x. \]