Limits

1. Let \( f(x) \) be a function such that
   - domain \((f) = [0,1).\)
   - For all \( x \) (in \([0,1]\)), \( 0 \leq f(x) < 1.\)
   - The function \( f \) is increasing on \([0,1).\)

   Show that there is a number \( L, 0 \leq L \leq 1, \) such that
   \[
   \lim_{x \to 1^-} f(x) = L.
   \]

   **Hint:** Construct a binary expansion for \( L.\)

2. Discuss the continuity of the function described on p. 97 and whose graph is sketched in FIGURE 14.

3. Prove: If \( g \) is continuous at \( a, g(a) \neq 0, \) then there is a \( \delta > 0 \) for which \((a - \delta, a + \delta)\) is contained in the domain of \( 1/g.\)

   **Solution.** For every \( \epsilon > 0, \) there is some \( \delta > 0 \) such that, for all \( x, \) if \( |x - a| < \delta, \) then \( |g(x) - g(a)| < \epsilon.\)

   Let \( \epsilon = |g(a)|. \) Then there is a \( \delta > 0 \) such that for \( |x - a| < \delta, \) \( |g(x) - g(a)| < |g(a)|. \) Thus for \( a - \delta < x < a + \delta, g(a) - |g(a)| < g(x) < g(a) + |g(a)|; \) if \( g(a) > 0, 0 < g(x) < 2g(a); \) if \( g(a) < 0, 2g(a) < g(x) < 0. \) In either case, for \( a - \delta < x < a + \delta, g(x) \neq 0, \) and \( x \) is in the domain of \( 1/g.\)

   **Another Solution.** For every \( \epsilon > 0, \) there is some \( \delta > 0 \) such that, for all \( x, \) if \( |x - a| < \delta, \) then \( |g(x) - g(a)| < \epsilon.\)

   Let \( \epsilon = |g(a)|. \) Then there is a \( \delta > 0 \) such that for \( |x - a| < \delta, \) \( |g(x) - g(a)| < |g(a)|. \) Thus for \( a - \delta < x < a + \delta, |g(x)| = |g(a) + (g(x) - g(a))| \geq |g(a) - |g(x) - g(a)|| > 0. \) Here we have used the triangle inequality in the form \( |A \pm B| \geq |A| - |B|. \)

   Thus, for \( a - \delta < x < a + \delta, g(x) \neq 0, \) and \( x \) is in the domain of \( 1/g.\)

   **Good Variation ... \( \epsilon = |g(a)| ... \) If \( g(a) > 0, ... \) for \( a - \delta < x < a + \delta, g(x) \in (g(a) - \epsilon, g(a) + \epsilon) = (0, 2g(a)) \) and \( g(x) \neq 0. ... \)