Real Numbers and Binary Expansions

The real numbers in \( \mathbb{R} \) are identified with points on a horizontal line. For the time being, we will identify a real number \( x \) with a decimal expansion.

- Every decimal expansion represents a real number \( x \):
  \[
  x = \pm N.d_1d_2\ldots, \\
  d_k \in \{0, 1, \ldots, 9\}.
  \]
  This is the statement that every infinite series of the form
  \[
  d_110^{-1} + d_210^{-2} + \ldots, \quad d_k \in \{0, 1, \ldots, 9\},
  \]
  converges.
  
  Just as well we could identify a real number \( x \) with a binary expansion.

- Every binary expansion represents a real number \( x \):
  \[
  x = \pm N_.binb_1b_2\ldots, \\
  b_k \in \{0, 1\}.
  \]
  This is the statement that every infinite series of the form
  \[
  b_12^{-1} + b_22^{-2} + \ldots, \quad b_k \in \{0, 1\},
  \]
  converges.

A demonstration of a correspondence between the binary expansion and a point on a horizontal line was given in class.

**Constructing the Binary Expansion**

Let \( x \) be a real number, \( 0 \leq x < 1 \).

We divide \([0, 1)\) into two half-open intervals, \([0, \frac{1}{2})\) and \(\left[\frac{1}{2}, 1\right)\).
Notice, that in binary notation we may write the two intervals as \([0, \text{bin} 1)\) and \([\text{bin} 1, 1)\).

If \(x\) is in the left interval, \(x = 0 + 0 \cdot 2^{-1} + \ldots\), so we let \(b_1 = 0, s_1 = 0 \cdot \text{bin} b_1\), so that
\[
x = 0 \cdot \text{bin} 0 + \ldots
= s_1 + r_1,
\]
where
\[
0 \leq r_1 < 2^{-1}.
\]

If \(x\) is in the right interval, \(x = 0 + 1 \cdot 2^{-1} + \ldots\), so we let \(b_1 = 1, s_1 = 0 \cdot \text{bin} b_1\), so that
\[
x = 0 \cdot \text{bin} 1 + \ldots
= s_1 + r_1,
\]
where
\[
0 \leq r_1 < 2^{-1}.
\]

We formalize the process by saying: Let
\[
b_1 = \begin{cases} 
0, & 0 \leq x < \frac{1}{2}, \\
1, & \frac{1}{2} \leq x < 1,
\end{cases}
\]
\[
s_1 = 0.\text{bin} b_1
= b_1 \cdot 2^{-1},
\]
\[
x = s_1 + r_1,
\]
\[
0 \leq r_1 < 2^{-1}.
\]

If the first remainder \(r_1\) is 0, that is \(x = 0\) or \(x = \frac{1}{2}\), STOP! – \(x = s_1\) and the binary expansion of \(x\) has been found.

If \(r_1 \neq 0\), we apply a similar process to \(r_1 = x - s_1\) to find the second binary digit in the expansion of \(x\). Let \(r_1^* = 2^1 \cdot r_1\). Once again divide \([0, 1)\) into the two half-open intervals, \([0, \frac{1}{2})\) and \([\frac{1}{2}, 1)\).
If $r^*_1$ is in the left interval, $x = 0 + b_1 \cdot 2^{-1} + 0 \cdot 2^{-2} + \ldots$. If $r^*_1$ is in the right interval, $x = 0 + b_1 \cdot 2^{-1} + 1 \cdot 2^{-2} + \ldots$, so we let $b_1 = 1$. This is same is saying that $x$ is in the left or right half as the interval selected in step 1.

Thus

$$b_2 = \begin{cases} 0, & 0 \leq r^*_1 < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r^*_1 < 1, \end{cases}$$

$$s_2 = 0_{\text{bin}} b_1 b_2$$

$$= s_1 + b_2 \cdot 2^{-2}$$

$$x = s_2 + r_2,$$

$$0 \leq r_2 < 2^{-2}.$$

If the second remainder $r_2$ is 0, STOP! – $x = s_2$ and the binary expansion of $x$ has been found. Otherwise continue!

The continuation may be defined by the Principle of Mathematical Induction (Recursion) so that if $b_n, s_n = 0_{\text{bin}} b_1 \ldots b_n, x = s_n + r_n$, have been constructed so that $0 \leq r_n < 2^{-n}$, let

$$r^*_n = 2^n \cdot r_n,$$

$$b_{n+1} = \begin{cases} 0, & 0 \leq r^*_n < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r^*_n < 1, \end{cases}$$

$$s_{n+1} = 0_{\text{bin}} b_1 b_2 \ldots b_n b_{n+1}$$

$$= s_n + b_{n+1} \cdot 2^{-(n+1)}$$

$$x = s_{n+1} + r_{n+1},$$

$$0 \leq r_{n+1} < 2^{-(n+1)}.$$

If the remainder $r_{n+1}$ is 0, STOP! – $x = s_{n+1}$ and the binary expansion of $x$ has been found. Otherwise continue!

**What has been done:** Given $x, 0 \leq x < 1$, there is a sequence $\{s_n\}_{n=1}^{\infty}$ of finite binary expansions such that $0 \leq x - s_n < 2^{-n}$. 
Binary Expansion Arguments

Consider the following problem:

1. Let \( f(x) \) be a function such that
   - domain \( (f) = [0, 1) \).
   - For all \( x \) (in \( [0, 1) \)), \( 0 \leq f(x) < 1 \).
   - The function \( f \) is increasing on \([0, 1)\).

Show that there is a number \( L \), \( 0 \leq L \leq 1 \), such that
   \[
   \lim_{x \to 1^-} f(x) = L.
   \]

**Hint:** Construct a binary expansion for \( L \).

A picture is helpful! See [http://www.math.uic.edu/~lewis/math430/chap6d.pdf](http://www.math.uic.edu/~lewis/math430/chap6d.pdf)

To find the expansion for \( L \), ask the question:

Is there an \( x \in [0, 1) \) such that \( f(x) \geq \frac{1}{2} = 0.\text{bin} 1 \)?

If NO, let \( x_1 = 0, b_1 = 0, s_1 = 0.\text{bin} b_1 \). If YES, let \( x_1 = x, b_1 = 1, s_1 = 0.\text{bin} b_1 = 0.\text{bin} 1 \). In both cases, for \( x_1 \leq x < 1, s_1 \leq f(x) \leq f(x) \leq s_1 + \frac{1}{2} \).

Next divide the interval \([s_1, s_1 + \frac{1}{2}]\) into two parts \([0.\text{bin} b_10, 0.\text{bin} b_11]\) and \([0.\text{bin} b_11, s_1 + \frac{1}{2}]\).

Ask the question:

Is there an \( x \in [x_1, 1) \) such that \( f(x) \geq s_1 + \frac{1}{2} = 0.\text{bin} b_11 \)?

If NO, let \( x_2 = x_1, b_2 = 0, s_2 = 0.\text{bin} b_1 b_2 \). If YES, let \( x_2 = x, b_1 = 1, s_2 = 0.\text{bin} b_1 b_2 = s_1 + \frac{1}{2} \). Then for \( x_2 \leq x < 1, s_2 \leq f(x) \leq f(x) \leq s_2 + \frac{1}{2} \).

If \( x_1, \ldots, x_n, b_1, \ldots b_n, s_n = 0.\text{bin} b_1 \ldots b_n \) have been constructed so that for \( x_n \leq x < 1, s_n \leq f(x_n) \leq f(x) \leq s_n + \frac{1}{2^n} \).

Ask the question: Is there an \( x \in [x_n, 1) \) such that \( f(x_{n+1}) \geq s_n + \frac{1}{2^{n+1}} = 0.\text{bin} b_1 \ldots b_n 1 \)?

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1 Thinking about this later, I noticed that \( s_1 + \frac{1}{2^2} \) is the midpoint of the new interval under consideration.
Then let

\[ x_{n+1} = \begin{cases} 
  x_n, & \text{NO}, \\
  x, & \text{YES},
\end{cases} \]

\[ b_{n+1} = \begin{cases} 
  0, & \text{NO}, \\
  1, & \text{YES},
\end{cases} \]

\[ s_{n+1} = 0 \text{bin} b_1 b_2 \ldots b_{n+1} = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_{n+1} \cdot 2^{-(n+1)} \]

For \( x_{n+1} \leq x < 1 \), \( s_{n+1} \leq f(x_{n+1}) \leq f(x) \leq s_{n+1} + \frac{1}{2^{n+1}} \).

Then

\[ L = 0.b_1 b_2 \ldots b_n \ldots = \lim_{n \to \infty} s_n \]

since for all \( x \), if \( x \leq x < 1 \),

\[ s_n \leq f(x) \leq L \leq s_n + \frac{1}{2^n} \]

and

\[ 0 \leq L - f(x) < \frac{1}{2^n}. \]

This shows that

\[ \lim_{x \to 1^-} f(x) = L. \]

**Additional Arguments Constructing the Binary Expansion of the Number Sought**