MthT 430 Notes Chapter 6c Binary Expansions

Real Numbers and Binary Expansions

The real numbers in $\mathbf{R}$ are identified with points on a horizontal line. For the time being, we will identify a real number x with a *decimal expansion*.

- Every decimal expansion represents a real number x:

$$x = \pm N.d_1 d_2 \ldots,$$
$$d_k \in \{0, 1, ..., 9\}.$$

This is the statement that every infinite series of the form

$$d_1 10^{-1} + d_2 10^{-2} + \ldots, \quad d_k \in \{0, 1, ..., 9\},$$

converges.

Just as well we could identify a real number x with a *binary expansion*.

- Every binary expansion represents a real number x:

$$x = \pm N ._{\text{bin}} b_1 b_2 \ldots,$$
$$b_k \in \{0, 1\}.$$

This is the statement that every infinite series of the form

$$b_1 2^{-1} + b_2 2^{-2} + \ldots, \quad b_k \in \{0, 1\},$$

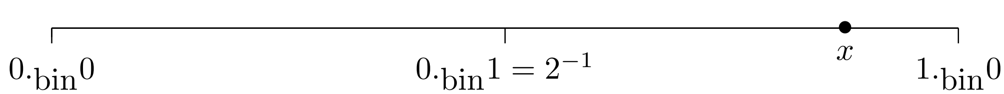
converges.

A demonstration of a correspondence between the binary expansion and a point on a horizontal line was given in class.

Constructing the Binary Expansion

Let x be a real number, $0 \leq x < 1$.

We divide $[0, 1)$ into two *half–open* intervals, $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$.



Notice, that in binary notation we may write the two intervals as $\left[0, 0._{\text{bin}}1\right)$ and $\left[0._{\text{bin}}1, 1\right)$.

If x is in the left interval, $x = 0 + 0 \cdot 2^{-1} + \ldots$, so we let $b_1 = 0$, $s_1 = 0._{\text{bin}}b_1$, so that

$$\begin{aligned} x &= 0._{\text{bin}}0 + \ldots \\ &= s_1 + r_1, \end{aligned}$$

where

$$0 \le r_1 < 2^{-1}.$$

If x is in the right interval, $x = 0 + 1 \cdot 2^{-1} + \ldots$, so we let $b_1 = 1$, $s_1 = 0._{\text{bin}}b_1$, so that

$$\begin{aligned} x &= 0._{\text{bin}}1 + \ldots \\ &= s_1 + r_1, \end{aligned}$$

where

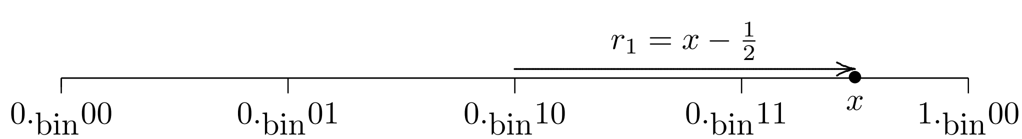
$$0 \le r_1 < 2^{-1}.$$

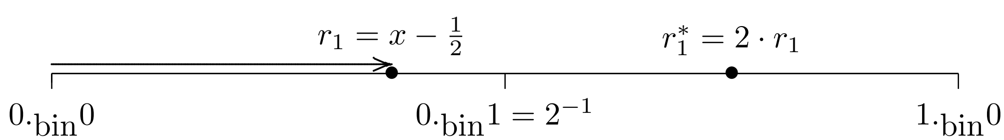
We formalize the process by saying: Let

$$\begin{aligned} b_1 &= \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x < 1, \end{cases} \\ s_1 &= 0._{\text{bin}}b_1 \\ &= b_1 \cdot 2^{-1}, \\ x &= s_1 + r_1, \\ 0 &\le r_1 < 2^{-1}. \end{aligned}$$

If the first *remainder* r_1 is 0, that is $x = 0$ or $x = \frac{1}{2}$, STOP! – $x = s_1$ and the binary expansion of x has been found.

If $r_1 \neq 0$, we apply a similar process to $r_1 = x - s_1$ to find the second binary digit in the expansion of x. Let $r_1^* = 2^1 \cdot r_1$. Once again divide $[0, 1)$ into the two *half–open* intervals, $\left[0, \dfrac{1}{2}\right)$ and $\left[\dfrac{1}{2}, 1\right)$.





If r_1^* is in the left interval, $x = 0 + b_1 \cdot 2^{-1} + 0 \cdot 2^{-2} + \ldots$. If r_1^* is in the right interval, $x = 0 + b_1 \cdot 2^{-1} + 1 \cdot 2^{-2} + \ldots$, so we let $b_1 = 1$. This is same is saying that x is in the left or right half as the interval selected in step 1.

Thus

$$\begin{aligned} b_2 &= \begin{cases} 0, & 0 \leq r_1^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r_1^* < 1, \end{cases} \\ s_2 &= 0._{\text{bin}} b_1 b_2 \\ &= s_1 + b_2 \cdot 2^{-2} \\ x &= s_2 + r_2, \\ 0 &\leq r_2 < 2^{-2}. \end{aligned}$$

If the second remainder r_2 is 0, STOP! – $x = s_2$ and the binary expansion of x has been found. Otherwise continue!

The continuation may be defined by the Principle of Mathematical Induction (Recursion) so that if b_n, $s_n = 0._{\text{bin}} b_1 \ldots b_n$, $x = s_n + r_n$, have been constructed so that $0 \leq r_n < 2^{-n}$, let

$$\begin{aligned} r_n^* &= 2^n \cdot r_n, \\ b_{n+1} &= \begin{cases} 0, & 0 \leq r_n^* < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq r_n^* < 1, \end{cases} \\ s_{n+1} &= 0._{\text{bin}} b_1 b_2 \ldots b_n b_{n+1} \\ &= s_n + b_{n+1} \cdot 2^{-(n+1)} \\ x &= s_{n+1} + r_{n+1}, \\ 0 &\leq r_{n+1} < 2^{-(n+1)}. \end{aligned}$$

If the remainder r_{n+1} is 0, STOP! – $x = s_{n+1}$ and the binary expansion of x has been found. Otherwise continue!

What has been done: Given x, $0 \leq x < 1$, there is a sequence $\{s_n\}_{n=1}^{\infty}$ of finite binary expansions such that $0 \leq x - s_n < 2^{-n}$.

Binary Expansion Arguments

Consider the following problem:

1. Let $f(x)$ be a function such that

- domain $(f) = [0, 1)$.
- For all x (in $[0, 1)$), $0 \le f(x) < 1$.
- The function f is increasing on $[0, 1)$.

Show that there is a number L, $0 \le L \le 1$, such that

$$\lim_{x \to 1^-} f(x) = L.$$

Hint: Construct a binary expansion for L.

A picture is helpful! See http://www.math.uic.edu/ lewis/math430/chap6d.pdf

To find the expansion for L, ask the question:

Is there an $x \in [0, 1)$ such that $f(x) \ge \frac{1}{2} = 0._{\text{bin}}1$?

If NO, let $x_1 = 0$, $b_1 = 0$, $s_1 = 0._{\text{bin}}b_1$. If YES, let $x_1 = x$, $b_1 = 1$, $s_1 = 0._{\text{bin}}b_1 = 0._{\text{bin}}1$. In both cases, for $x_1 \le x < 1$, $s_1 \le f(x_1) \le f(x) \le s_1 + \frac{1}{2}$.

Next divide the interval $\left[s_1, s_1 + \frac{1}{2}\right)$ into two parts $\left[0._{\text{bin}}b_1 0, 0._{\text{bin}}b_1 1\right)$ and $\left[0._{\text{bin}}b_1 1, s_1 + \frac{1}{2}\right)$. Ask the question:

Is there an $x \in [x_1, 1)$ such that $f(x) \ge s_1 + \dfrac{1}{2^2} = 0._{\text{bin}}b_1 1$?[1]

If NO, let $x_2 = x_1$, $b_2 = 0$, $s_2 = 0._{\text{bin}}b_1 b_2$. If YES, let $x_2 = x$, $b_1 = 1$, $s_2 = 0._{\text{bin}}b_1 b_2 = s_1 + \dfrac{1}{2^2}$. Then for $x_2 \le x < 1$, $s_2 \le f(x_2) \le f(x) \le s_2 + \dfrac{1}{2^2}$.

If $x_1, \ldots, x_n$, $b_1, \ldots b_n$, $s_n = 0._{\text{bin}}b_1 \ldots b_n$ have been constructed so that for $x_n \le x < 1$, $s_n \le f(x_n) \le f(x) \le s_n + \dfrac{1}{2^n}$,

Ask the question: Is there an $x \in [x_n, 1)$ such that $f(x_{n+1}) \ge s_n + \dfrac{1}{2^{n+1}} = 0._{\text{bin}}b_1 \ldots b_n 1$?

---

[1] Thinking about this later, I noticed that $s_1 + \dfrac{1}{2^2}$ is the *midpoint* of the new interval under consideration.

Then let

$$
\begin{aligned}
x_{n+1} &= \begin{cases} x_n, & \text{NO}, \\ x, & \text{YES}, \end{cases} \\
b_{n+1} &= \begin{cases} 0, & \text{NO}, \\ 1, & \text{YES}, \end{cases} \\
s_{n+1} &= 0._{\text{bin}} b_1 b_2 \ldots b_{n+1} \\
&= b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_{n+1} \cdot 2^{-(n+1)}
\end{aligned}
$$

$$
\text{For } x_{n+1} \leq x < 1, \quad s_{n+1} \leq f(x_{n+1}) \leq f(x) \leq s_{n+1} + \frac{1}{2^{n+1}}.
$$

Then

$$
L = 0.b_1 b_2 \ldots b_n \ldots = \lim_{n \to \infty} s_n
$$

since for all x, if $x_n \leq x < 1$,

$$
s_n \leq f(x) \leq L \leq s_n + \frac{1}{2^n}
$$

and

$$
0 \leq L - f(x) < \frac{1}{2^n}.
$$

This shows that

$$
\lim_{x \to 1^-} f(x) = L.
$$

Additional Arguments Constructing the Binary Expansion of the Number Sought

See `http://www.math.uic.edu/~lewis/mtht430/chap7b.pdf`