MthT 430 Projects Chapter 8d Adding sup and inf

It is useful to note a working characterization of $\sup A$ $(A \neq \emptyset)$:

If $A \neq \emptyset$, $\sup A$ is the number α such that

$$\begin{cases} \text{For every } x \in A,\ x \leq \alpha, \text{ and} \\ \text{For every } \epsilon > 0, \text{ there is an } x \in A \text{ such that } x > \alpha - \epsilon. \end{cases}$$

The first condition means that α *is an upper bound for* A. The second condition means *for every* $\epsilon > 0$, $\alpha - \epsilon$ *is not an upper bound for* A.

1. Show that if x and y are numbers, then $x \leq y$ if and only if for every $\epsilon > 0$, $x < y + \epsilon$.

2. Let A be a nonempty set of numbers which is bounded above. Show that

$$b = \sup A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x < b + \epsilon & \text{for all } x \in A, \text{ and} \\ x > b - \epsilon & \text{for some } x \in A. \end{cases}$$

3. Let A be a nonempty set of numbers which is bounded below. Show that

$$b = \sup A$$

if and only if for every $\epsilon > 0$

$$\begin{cases} x > b - \epsilon & \text{for all } x \in A, \text{ and} \\ x < b + \epsilon & \text{for some } x \in A. \end{cases}$$

4. (See Chapter 8 – Problem 13) Let A and B be two nonempty sets of numbers which are bounded (both above and below). Define

$$A + B = \{x \mid x = a + b, a \in A, b \in B\}.$$

Show that

$$\sup(A + B) = \sup A + \sup B.$$

Show that

$$\inf(A + B) = \inf A + \inf B.$$