MthT 430 Chap 8h $\limsup$ and $\liminf$ for Functions

- (See also Spivak Chapter 8 - Problem 18) Let $\{x_k\}$ be a bounded sequence. We define the *limit superior* and *limit inferior* of the sequence to be

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty}\left(\sup_{n\ge k} x_n\right),$$
$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty}\left(\inf_{n\ge k} x_n\right).$$

For the time being, we speak of a function $f(x)$ defined and bounded near $x = a$. In the same spirit, we define $\limsup$ and $\liminf$ for functions.

- Let f be a bounded function. We define the *limit superior* and *limit inferior* of f near a to be

$$\limsup_{x\to a} f(x) = \lim_{\delta\to 0^+}\left(\sup_{0<|x-a|<\delta} f(x)\right),$$
$$\liminf_{x\to a} f(x) = \lim_{\delta\to 0^+}\left(\inf_{0<|x-a|<\delta} f(x)\right).$$

- (P13–LUB) shows that both $\limsup_{x\to a} f(x)$ and $\liminf_{x\to a} f(x)$ exist.

Definition. *If f is a bounded function defined near a, we define*

$$M(f,a,\delta) = \sup_{0<|x-a|<\delta} f(x),$$
$$m(f,a,\delta) = \inf_{0<|x-a|<\delta} f(x).$$

For $\delta > 0$ and small enough, $M(f,a,\delta)$ is a bounded nondecreasing function of δ, and

$$\lim_{\delta\to 0^+} M(f,a,\delta) = \inf_{\delta>0} M(f,a,\delta)$$

exists.

For $\delta > 0$ and small enough, $m(f,a,\delta)$ is a bounded nonincreasing function of δ, and

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exists.

Necessary and Sufficient Condition (NASC) for Existence of a Limit of a Function

- Show that
$$\limsup_{x\to a} f(x) = A$$
if and only if for every $\epsilon > 0$,
$$\begin{cases} \text{there is a } \delta > 0 \text{ such that } f(x) < A + \epsilon \text{ for } 0 < |x-a| < \delta, \\ \text{for } \textbf{every } \delta > 0, \text{ there is an } x_\delta,\ 0 < |x_\delta - a| < \delta, \text{ such that } f(x_\delta) > A - \epsilon. \end{cases}$$

- Show that
$$\liminf_{x\to a} f(x) = A$$
if and only if for every $\epsilon > 0$,
$$\begin{cases} \text{there is a } \delta > 0 \text{ such that } f(x) > A - \epsilon \text{ for } 0 < |x-a| < \delta, \\ \text{for } \textbf{every } \delta > 0, \text{ there is an } x_\delta,\ 0 < |x_\delta - a| < \delta, \text{ such that } f(x_\delta) < A + \epsilon. \end{cases}$$

- Prove:

Theorem. *Let f be a bounded function for x near a and $\neq a$. Then*
$$\lim_{x\to a} f(x) \text{ exists}$$
if and only if
$$\liminf_{x\to a} f(x) = \limsup_{x\to a} f(x)$$

Proof. Easy - If $\lim_{x\to a} f(x) = L$, then $\liminf_{x\to a} f(x) = L, \ldots$. For the converse, use the first of the two conditions in the characterizations of $\limsup_{x\to a} f(x) = L$ and $\liminf_{x\to a} f(x) = L$.