Let \( \{x_k\} \) be a bounded sequence. We define the \textit{limit superior} and \textit{limit inferior} of the sequence to be

\[
\limsup_{k \to \infty} x_k = \lim_{k \to \infty} \left( \sup_{n \geq k} x_n \right),
\]
\[
\liminf_{k \to \infty} x_k = \lim_{k \to \infty} \left( \inf_{n \geq k} x_n \right).
\]

For the time being, we speak of a function \( f(x) \) defined and bounded near \( x = a \). In the same spirit, we define \( \limsup \) and \( \liminf \) for functions.

Let \( f \) be a bounded function. We define the \textit{limit superior} and \textit{limit inferior} of \( f \) near \( a \) to be

\[
\limsup_{x \to a} f(x) = \lim_{\delta \to 0^+} \left( \sup_{0 < |x-a| < \delta} f(x) \right),
\]
\[
\liminf_{x \to a} f(x) = \lim_{\delta \to 0^+} \left( \inf_{0 < |x-a| < \delta} f(x) \right).
\]

(P13–LUB) shows that both \( \limsup_{x \to a} f(x) \) and \( \liminf_{x \to a} f(x) \) exist.

\textbf{Definition.} \textit{If} \( f \) \textit{is a bounded function defined near} \( a \), \textit{we define}

\[
M(f, a, \delta) = \sup_{0 < |x-a| < \delta} f(x),
\]
\[
m(f, a, \delta) = \inf_{0 < |x-a| < \delta} f(x).
\]

For \( \delta > 0 \) and small enough, \( M(f, a, \delta) \) is a bounded nondecreasing function of \( \delta \), and

\[
\lim_{\delta \to 0^+} M(f, a, \delta) = \inf_{\delta > 0} M(f, a, \delta)
\]

exists.

For \( \delta > 0 \) and small enough, \( m(f, a, \delta) \) is a bounded nonincreasing function of \( \delta \), and

\[
\lim_{\delta \to 0^+} m(f, a, \delta) = \sup_{\delta > 0} m(f, a, \delta)
\]

exists.
Necessary and Sufficient Condition (NASC) for Existence of a Limit of a Function

• Show that \( \limsup_{x \to a} f(x) = A \)
  if and only if for every \( \epsilon > 0 \),
  \[
  \begin{cases}
  \text{there is a } \delta > 0 \text{ such that } f(x) < A + \epsilon \text{ for } 0 < |x - a| < \delta, \\
  \text{for every } \delta > 0, \text{ there is an } x_\delta, 0 < |x_\delta - a| < \delta, \text{ such that } f(x_\delta) > A - \epsilon.
  \end{cases}
  \]

• Show that \( \liminf_{x \to a} f(x) = A \)
  if and only if for every \( \epsilon > 0 \),
  \[
  \begin{cases}
  \text{there is a } \delta > 0 \text{ such that } f(x) > A - \epsilon \text{ for } 0 < |x - a| < \delta, \\
  \text{for every } \delta > 0, \text{ there is an } x_\delta, 0 < |x_\delta - a| < \delta, \text{ such that } f(x_\delta) < A + \epsilon.
  \end{cases}
  \]

• Prove:

\textbf{Theorem.} Let \( f \) be a bounded function for \( x \) near \( a \) and \( \neq a \). Then
  \[
  \lim_{x \to a} f(x) \text{ exists}
  \]
  if and only if
  \[
  \liminf_{x \to a} f(x) = \limsup_{x \to a} f(x)
  \]

\textbf{Proof.} Easy - If \( \lim_{x \to a} f(x) = L \), then \( \liminf_{x \to a} f(x) = L \), .... For the converse, use the first of the two conditions in the characterizations of \( \limsup_{x \to a} f(x) = L \) and \( \liminf_{x \to a} f(x) = L \).