MthT 430 Chap 8XC Extra Credit for Really Understanding sup and inf

Extra Credit for Really Understanding sup and inf

8XC (See also Spivak Chapter 8 - Problem 18) Let $\{x_k\}$ be a bounded sequence. We define the *limit superior* ($\limsup$) and *limit inferior* ($\liminf$) of the sequence to be

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty}\left(\sup_{n\geq k} x_n\right),$$
$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty}\left(\inf_{n\geq k} x_n\right).$$

- Show that

  $$\limsup_{k\to\infty} x_k = A$$

  if and only if for every $\epsilon > 0$,

  $$\begin{cases} x_k > A+\epsilon & \text{for at most finitely many } k, \\ x_k > A-\epsilon & \text{for infinitely many } k. \end{cases}$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

  $$\limsup_{k\to\infty}(x_k+y_k) \leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Give an example of bounded sequences $\{x_k\}$ and $\{y_k\}$ such that

  $$\limsup_{k\to\infty}(x_k+y_k) < \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

  $$\liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \leq \limsup_{k\to\infty}(x_k+y_k).$$

The general result is that for two bounded sequences $\{x_k\}$ and $\{y_k\}$,

$$\begin{aligned} \liminf_{k\to\infty} x_k + \liminf_{k\to\infty} y_k &\leq \liminf_{k\to\infty}(x_k+y_k) \\ &\leq \liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \\ &\leq \limsup_{k\to\infty}(x_k+y_k) \\ &\leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k. \end{aligned}$$