MthT 430 Chapter 9a Spivak Problem Remarks

7. $f(x) = x^3$. $f'(x) = 3x^2$

(a) $f'(9) = 3 \cdot 9 = 27$.

(b) $f'(3^2) = f'9 = 27$ or $f'(3^2) = 3 \cdot \left(3^2\right)^2 = 27$.

(c) $f'(a^2) = 3 \cdot \left(a^2\right)^2 = 3a^4$; $f'(x^2) = 3 \cdot \left(x^2\right)^2 = 3x^4$.

(d) $f(x) = x^3$; $f'(x) = 3x^2$; $f'x^2 = 3x^4$. $g(x) = f(x^2) = x^6$; $g'(x) = 6x^5$.

8.

(a) $g(x) = f(x + c)$.

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+c+h) - f(x+c)}{h} \\ &= f'(x+c). \end{aligned}$$

(b) $g(x) = f(cx)$. For $c \neq 0$,

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(c(x+h)) - f(cx)}{h} \\ &= \lim_{ch \to 0} c \cdot \frac{f(cx+ch)0 - f(cx)}{ch} \\ &= c \cdot f'(cx). \end{aligned}$$

10. $f(x) = g(t + x)$. $f'(a) = g'(t + a)$; $f'(x) = g'(t + x)$.

$F(t) = g(t + x)$. $F'(a) = g'(a + x)$; $F'(x) = g'(x + x)$.

11.

(a) If s' is proportional to s, there is a constant k such that $s'(t) = ks(t)$. For $s(t) \neq 0$, $s'(t)/s(t)$ is constant.

If $S(t) = ct^2$, $S'(t) = 2ct$. For $t \neq 0$, $S(t) \neq 0$, and $S'(t)/S(t) = 2/t$, which is not constant.

(b) If $s(t) = (a/2)\, t^2$,

$$s'(t) = at.$$
$$s''(t) = a.$$

Note that

$$\begin{aligned}(s'(t))^2 &= (at)^2 \\ &= 2as(t).\end{aligned}$$

12. Speed limit at *position* x is $L(x)$. Position of A at *time* t is denoted by $a(t)$.

(a) A travels at the speed limit means: For all t, $a'(t) = L(a(t))$.

(b) Suppose A travels at the speed limit and $b(t) = a(t-1)$. Then $b'(t) = a'(t-1) = L(a(t-1)) = L(b(t))$, and B travels at the speed limit.

(c) If $b(t) = a(t) - k$, $b'(t) = a'(t) = L(a(t))$. Then $b'(t) = L(b(t))$ for all t, if and only if $L(b(t)) = L(a(t) - k) = L(a(t))$, or $L(x)$ is periodic with period k.

18. f is the *oneoverq* function. If r is a rational number, f is not continuous at r. Thus f is not differentiable at r.

If a is an irrational number, $f(a) = 0$. If h is rational, the difference quotient is 0. Thus if $f'(a)$ exists, $f'(a) = 0$.

Let a have the nonrepeating decimal expansion $m.a_1a_2 \ldots a_n \ldots$. Define the irrational number $h_n = -0.00 \ldots 0a_na_{n+1} \ldots$, so that $a + h_n = m.a_1a_2 \ldots a_{n-1}$.

Now $|h_n| \le 10^{1-n}$, $|1/h_n| \ge 10^{n-1}$, and $f(a + h_n) = 1/q$ with $q \le 10^{n-1}$ so that $|f(a + h_n)| = 1/q \ge 10^{1-n}$.

It follows that

$$\begin{aligned}\left|\frac{f(a+h_n) - f(a)}{h_n}\right| &= \frac{|f(a+h_n)|}{|h_n|} \\ &= \frac{1/q}{|h_n|} \ge 10^{1-n}10^{n-1} = 1.\end{aligned}$$

Conclude that $f'(a)$ does not exist.