MthT 430 Notes Chapter 9 Limits and Order

Limits and Order

For functions of a real variable, the derivative is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

which means that the difference

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x)$$

is *small* if Δx is small and not 0 (for which the quotient is not obviously defined).

Multiplying the remainder by Δx, we obtain that

$$f(x+\Delta x) - f(x) - f'(x)\Delta x = \mathit{small} \cdot \Delta x,$$

with the right hand side (**RHS**) of the equation is "*much smaller than* Δx" in the precise sense

$$\lim_{\Delta x \to 0} \frac{\mathbf{RHS}}{|\Delta x|} = 0.$$

Another formal advantage is that the equation is also defined and true for $\Delta x = 0$.

Definition. An expression (function) $\phi(x)$ is *little o of x* as $x \to 0$, written $\phi(x) = o\,(x)$ [as $x \to 0$], if

$$\lim_{x \to 0} \frac{\phi(x)}{|x|} = 0.$$

If we are not worried about the particular details of $\phi(x)$, we write $\phi(x) = o\,(x)$ [as $x \to 0$].

With this convention, the definition of differentiability and the derivative takes the convenient form

$$f(x+\Delta x) = f(x) + f'(x) \cdot \Delta x + o\,(\Delta x)\,.$$

In a similar way, if $\lim_{x \to 0} \psi(x) = 0$, we write $\psi(x) = o\,(1)$ with the precise meaning that

$$\lim_{x \to 0} \frac{\psi(x)}{1} = 0.$$

Definition. *Let $q(x)$ be nonzero for x near and not equal 0. Then a function $\phi(x)$ is little o of $q(x)$, written $\phi(x) = o\,(q(x))$, if*

$$\lim_{x \to 0} \frac{\phi(x)}{|q(x)|} = 0.$$

Then a function $\phi(x)$ is big O of $q(x)$, written $\phi(x) = O\left(q(x)\right)$, if

$$\frac{\phi(x)}{|q(x)|}$$

is bounded as $x \to 0$.

N.B. We are assuming that, for x small enough and $\neq 0$, both $\phi(x)$ and $q(x)$ are defined and $q(x) \neq 0$.

We could also propose an ϵ–δ definition of *little* $o(q)$:

Definition. *A function $f = o(q)$ as $x \to 0$ means For every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x| < \delta$, then $|f(x)| < \epsilon|q(x)|$*

With this convention, continuity of a function $f(x)$ can be expressed by

$$f(x + \Delta x) = f(x) + o\left(1\right)$$

as $\Delta x \to 0$.

A real valued function of a real variable x is differentiable at x with derivative $f'(x)$ if

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + o\left(\Delta x\right)$$

as $\Delta x \to 0$.

Local boundedness of a function can be expressed as $f(x + \Delta x) = O\left(1\right)$ as $\Delta x \to 0$.

There is a formal calculus for handling sums and products for functions which are *little o* or *big O* of one (or several) q. Verify that $O\left(1\right)\cdot o\left(\Delta x\right) = o\left(\Delta x\right)$; i.e., the product of a bounded function and a function which is $o\left(\Delta x\right)$ is $o\left(\Delta x\right)$. Similarly $o\left(\Delta x\right) \pm o\left(\Delta x\right) = o\left(\Delta x\right)$.

Proof of the Chain Rule

The Chain Rule. Let $g(z)$ be differentiable at z, and let $f(w)$ be differentiable at $w = g(z)$. Then $h(z) = f(g(z))$ is differentiable at z and

$$h'(z) = \frac{d}{dz} f(g(z)) = f'\left(g(z)\right) \cdot g'(z).$$

Proof: We show that

$$\begin{aligned} h(z + \Delta z) &= f(g(z + \Delta z)) \\ &= f(g(z)) + f'(g(z)) \cdot g'(z) \cdot \Delta z + o\left(\Delta z\right). \end{aligned}$$

as $\Delta z \to 0$.

Let

$$\Delta g(z) = g(z + \Delta z) - g(z) = g'(z)\Delta z + o\left(\Delta z\right).$$

We are assuming that

$$\begin{aligned} g(z + \Delta z) &= g(z) + g'(z) \cdot \Delta z + o\left(\Delta z\right), \\ f(g(z) + \Delta g(z)) &= f(g(z)) + f'(g(z)) \cdot \Delta g(z) + o\left(\Delta g(z)\right). \end{aligned}$$

Since

$$\begin{aligned} \Delta g(z) &= g'(z)\Delta z + o\left(\Delta z\right) \\ &= O\left(\Delta z\right), \\ o\left(\Delta g(z)\right) &= o\left(O\left(\Delta z\right)\right) \\ &= o\left(\Delta z\right), \end{aligned}$$

we have

$$f(g(z + \Delta z)) = f(g(z)) + f'(g(z)) \cdot g'(z) \cdot \Delta z + o\left(\Delta z\right).$$

Remarks

The concepts *little* o and *big* O are also useful as the argument $x \to \infty$. For example we write $x^2 = o\left(e^x\right)$ as $x \to \infty$ with the precise meaning

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$