MthT 491 Mathematics of the Chipboard

Jeff E. Lewis

This note will follow one of the standard constructions[1] of the negative integers and its relation to the Chip Board representation and model for integers.

We shall assume that the natural numbers and 0 – the nonnegative integers – $N_0 = \{0, 1, 2, 3, \ldots\}$ are available and the binary operations of addition, +, and multiplication, $\cdot$ are defined so that

Properties of Addition

P1 For all a, b, c, in N_0,

$$a + (b + c) = (a + b) + c$$

P2 There is a number 0 in N_0 such that for all a,

$$a + 0 = 0 + a = a.$$

P3 (Not Yet!) [Additive Inverse]

P4 For all a, b,

$$a + b = b + a.$$

P5 For all a, b, c,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

P6 [Multiplicative Unit] There is a number $1 \neq 0$ such that for all a,

$$a \cdot 1 = 1 \cdot a = a.$$

P7 (Not Yet!) [Multiplicative Inverse]

P8 For all a, b,

$$a \cdot b = b \cdot a.$$

Property of $\cdot$ with +

P9 (Distributive) For all a, b, c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) = a \cdot b + a \cdot c.$$

[1] I thank John Wood for his explanation of the construction in the style of Grothendieck.

We have intentionally omitted property (P3) – the *additive inverse* – which holds for the set of all integers, but which does not hold for the set of nonnegative integers:

P3 **Not Yet!** For all a, there is a number $-a$ such that

$$a + (-a) = (-a) + a = 0.$$

For now, we also omit any mention of a *multiplicative inverse*:

P7 **Not Yet!** For all $a \neq 0$, there is a number a^{-1} such that

$$a \cdot (a^{-1}) = (a^{-1}) \cdot a = 1.$$

The aim is to give a formal construction which will enlarge the set of nonnegative integers to include the *negative integers*. We will do this by making a formal sense of $a - b$ for any $a \geq 0, b \geq 0$.

Think of $a - b$ as an ordered pair (a, b). There are many different ways (ordered pairs) which give the same result $a - b$: For any $c \geq 0$,

$$a - b = (a + c) - (b + c)\,.$$

We want to identify (treat as the same) the pair (a, b) with all the pairs $(a + c, b + c)$.

Motivated by the desired behavior of $a - b$,

$$a - b = a' - b'$$

iff

$$a + b' = a' + b,$$

we define an equivalence relation on ordered pairs (a, b).

Chip Board. *Think of the pair (a, b) as a Board with a black chips and b red chips. Adding c black chips and c red chips gives a Board with the same value.*

Definition. *Consider the collection of all ordered pairs (a, b), $a, b \in N_0$. We say that*

$$(a, b) \simeq (a', b')$$
$$\text{iff } a + b' = a' + b.$$

Read $(a, b) \simeq (a', b')$ as "The pair (a, b) is *equivalent* to the pair (a', b')."

Note that, for any $c \in N_0$,

$$(a, b) \simeq (a + c, b + c)\,.$$

We denote the equivalence class of (a, b) as $[a, b]$.

Chip Board. *For any $c \geq 0$, the board with a black chips and b red chips is considered the same (has the same value) as the board with $a + c$ black chips and $b + c$ red chips. Moreover, we can go between two equally valued boards by adjoining or taking away pairs of red and black chips.*

We can also think of the number a, $a \geq 0$, as corresponding to the Board with a black chips and 0 red chips:

$$a \leftrightarrow (a, 0)$$
$$\text{or } a \leftrightarrow [a, 0]\,.$$

Chip Board. *The Chip Board with 0 black and b red corresponds to the pair $(0, b)$ and also to $(c, b + c)$; thus 0 black and b red corresponds to the equivalence class $[0, b]$.*

We must verify that $\simeq$, is an *equivalence relation*, i.e., the relation $\simeq$ is

- *reflexive*: $(a, b) \simeq (a, b)$,
- *symmetric*: if $(a, b) \simeq (a, b)$, then $(a', b') \simeq (a, b)$,
- *transitive*: if $(a, b) \simeq (a', b')$, and $(a', b') \simeq (a'', b'')$ then $(a, b) \simeq (a'', b'')$.

Chip Board. *Call an* equivalence class *the* value *of any representative board.*

- *reflexive: Every Board has the same value as its own value.*
- *symmetric: if Board X has the same value as Board Y, then Board Y has the value of Board X.*
- *transitive: if Board X has the same value as Board Y, and Board Y has the same value as Board Z, then Board X has the same value as Board Z.*

We denote the equivalence class of (a, b) as $[a, b]$.

Next we define an addition and multiplication on equivalence classes, pushing toward making $[a, b]$ a pseudonym for $a - b$.

For the time being we shall indicate addition by $\oplus$ and multiplication by $\otimes$.

Addition

Indicate addition of equivalence classes by $\oplus$.

If $a \geq b \geq 0$, $c \geq d \geq 0$,

$$(a-b)+(c-d)=(a+c)-(b+d)\,.$$

Motivated by the desire to extend this property, for all $a, b, c, d \in N_0$, define

$$[a,b] \oplus [c,d] = [a+c, b+d]\,.$$

Chip Board. *Starting with a black and b red, put down c black and d red.*

Note that

- A **Chip Board** corresponding to (a,b) can be obtained by putting down a black chips and then putting down b red chips.
- $[a,b]=[a,0]\oplus[0,b]$.
- Any **Chip Board** with value 0 has an equal number of black and red chips – c black chips and c red chips. We call such a board the Null Board $\leftrightarrow [0,0]=[c,c]$.
- The Null Board $(0,0)$ serves as a *Zero* for addition $\oplus$,

  $$[a,b]\oplus[0,0]=[a,b]$$

- $[a,b]\oplus[b,a]=[a+b,a+b]=[0,0]$. Thus $[a,b]$ has an *additive inverse*, usually called $-[a,b]$, such that

  $$[a,b]+(-[a,b])=[0,0]\,.$$

- If the **Chip Board** with a black and b red is joined to the board with the Board with b black and a red constructs the Board with $a+b$ black and $a+b$ red – which is equivalent to the board with no chips (also called the Null Board).
- For $a, b \geq 0$, make the correspondences $a \leftrightarrow [a,0]\,, b \leftrightarrow [b,0]$. Then $(a+b)\leftrightarrow[a+b,0]=[a,0]\oplus[b,0]$.

Multiplication

Indicate multiplication of equivalence classes by $\otimes$.

Motivated by the desired property

$$(a-b)\cdot(c-d) = (ac+bd) - (bc+ad)\,,$$

define

$$[a,b]\otimes[c,d] \equiv [ac+bd, bc+ad]\,.$$

Chip Board. *Much more complicated. The desired distributive property has a major role.*

On the chip board, multiplying the Board with a black and b red by the number c, $c \geq 0$ will construct the Board with $a\cdot c$ black and $b\cdot c$ red. I.e.,

$$[a,b]\otimes[c,0] \equiv [ac, bc]\,.$$

Next, let us *agree* that multiplying by -1 should correspond to replacing every black chip by a red chip and every red chip by a black chip.† We should have that the Board with a black and b red, when multiplied by -1 should construct the board with b black and a red. Since

$$-1 \leftrightarrow (0,1)\,,$$
$$\text{or } -1 \leftrightarrow [0,1]\,,$$

we have

$$[a,b]\otimes[0,1] = [b,a]\,.$$

The general rule exploits the desired distributive property:

$$\begin{aligned}
[a,b]\otimes[c,d] &= [a,b]\otimes([c,0]\oplus[0,d]) \\
&= ([a,b]\otimes[c,0])\oplus([a,b]\otimes[0,d]) \\
&= ([a,b]\otimes[c,0])\oplus([a,b]\otimes(-[d,0])) \\
&= [ac,bc]\oplus[bd,ad] \\
&= [ac+bd, bc+ad]\,.
\end{aligned}$$

There is still work to be done. We must show that the operations $\oplus$ and $\otimes$ well defined on equivalence classes. This amounts to showing that

$$\begin{aligned}
\text{if } (a,b) &\simeq (a',b')\,, \\
\text{and } (c,d) &\simeq (c',c')\,, \\
\text{then } (a+c, b+d) &\simeq (a'+c', b'+d')\,, \\
\text{and } (ac+bd, bc+ad) &\simeq (a'c'+b'd', b'c'+a'd')\,.
\end{aligned}$$

† We try to force the rule $(-1)\,[a,b] = -\,[a,b]$

Note that, for any c,

$$[a,b]+[c,c]=[a,b]\,.$$

Thus the equivalence class $[0,0]=[c,c]$ serves as a zero for the addition $\oplus$.

Also $[1,0]=[c+1,c]$ serves the role of 1, the *unit* for $\otimes$:

$$\begin{aligned}[a,b]\otimes[1,0]&=[a\cdot 1,b]\\&=[a,b]\,.\end{aligned}$$

We identify the starting number a, $a\geq 0$, with its equivalence class $[a,0]=[a+c,c]$ (Think $a=(a+c)-c$.), and note that

$$[a+c,c]\oplus[b+d,d]=[a+b+(c+d)\,,c+d]\,,$$

which is identified with $a+b$.

Also

$$\begin{aligned}[a+c,c]\oplus[c,a+c]&=[a+c+c,a+c+c]\,,\\&=[c,c]\\&=\text{ Zero for }\oplus.\end{aligned}$$

$$\begin{aligned}[a,b]\oplus[b,a]&=[a+b,a+b]\,,\\&=\text{ Zero for }\oplus.\end{aligned}$$

Now if we identify 1 with $[1,0]=[c+1,c]$, then its additive inverse for $\oplus$ is $[0,1]=[c,c+1]$.

We can also check

$$\begin{aligned}[a,b]\otimes[0,1]&=[a\cdot 0+b\cdot 1,a\cdot 1+b\cdot 0]\\&=[b,a]\,.\end{aligned}$$

Thus it shown that

- The *additive inverse* of the unit $[1,0]$ is $[0,1]$,

$$-[1,0]=[0,1]\,.$$

- The *additive inverse* of $[a,b]$ is $[a,b]\otimes($ *additive inverse* of the unit $)$.

$$-[a,b]=(-[1,0])\otimes[a,b]\,.$$

-

Subtraction

Indicate subtraction of equivalence classes by $\ominus$.

Since

$$(a-b)-(c-d)=(a+d)-(b+c)\,,$$

define

$$\begin{aligned}[a,b]\ominus[c,d] &\equiv [a+d,b+c]\\ &=[a,b]\oplus[d,c]\\ &=[a,b]\oplus(-[c,d])\,.\end{aligned}$$

The defined subtraction, $\ominus$, now corresponds to the usual subtraction.

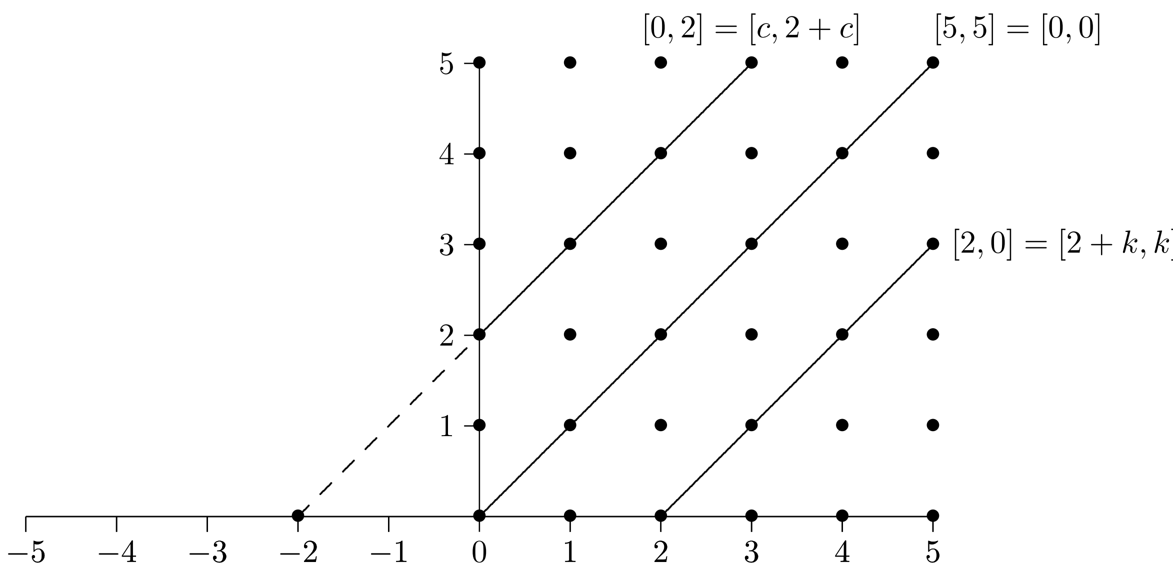
Notes

The same equivalence class construction of *negative numbers* could be carried out starting with the the nonnegative rational numbers or nonnegative real numbers. The role of the *chips* on the **Chip Board** might be played by colored lengths of string.

Geometric Interpretation of Equivalence Classes

Think of the pairs (a, b), $a, b \geq 0$ as points in the first quadrant. Pairs of the form $(a, 0)$, a black and no red, are on the nonnegative x–axis; pairs of the form $(0, b)$, no black and b red, are on the nonnegative y–axis.

A geometric representation of equivalence class of a pair (a, b) is all [integer] points in the first quadrant which are on the line of slope 1 through (a, b).



Note that if we start with $2 \leftrightarrow [2, 0]$, obtain its additive inverse $[0, 2]$, and then extend the line which represents $[0, 2]$, the line would intersect the x–axis at the point $(-2, 0)$.