MthT 491 Products of Negative Numbers

Example The Burning Candle

Consider the candle of length 18 cm which burns in 6 hrs. Then the rate of burning is

$$\frac{18 \text{ cm}}{6 \text{ hr}} = 3 \text{ cm/hr} \,.$$

Now if L is the length of that candle, L is changing at the rate of -3 cm/hr. Observe the candle when $L = 15$cm. Ask the question: What was the length of the candle 1 hour ago?

$$\begin{aligned} L &= 15 + (\text{rate}) \cdot (\text{time change}) \\ &= 15 + (-3) \cdot (-1) \\ &= 18. \end{aligned}$$

Thus $(-3) \cdot (-1) = +3$.

Discussion of Products

We need to think about the meaning of multiplication. However we interpret multiplication by positive integers, we should have a *distributive property*, e. g.

$$(3 + 2) \cdot a = 3 \cdot a + 2 \cdot a.$$

The trick is to interpret or "define" multiplication by negative numbers in a way that the distributive property is maintained. If we think of ${}^{+}1 \cdot a$ as adding a to whatever, we could define ${}^{-}1 \cdot a$ as "undoing" the addition, i.e. adding ${}^{-}a$ to whatever.

The crucial mathematical fact is that for a number a, the number ${}^{-}a$, *negative* or opposite of a is the same as the product $({}^{-}1) \cdot a$; i.e.,

$${}^{-}a = {}^{-}1 \cdot a.$$

The matter is rather delicate. If there is justice, we have the distributive property, $a\,(b + c) = ab + ac$. This tells us that

$$\begin{aligned} a + {}^{-}a &= 0 \\ &= 0 \cdot a \\ &= \big(1 + {}^{-}1\big) \cdot a \\ &= 1 \cdot a + {}^{-}1 \cdot a \\ &= a + {}^{-}1 \cdot a. \end{aligned}$$

It follows that

$${}^{-}a = {}^{-}1 \cdot a.$$