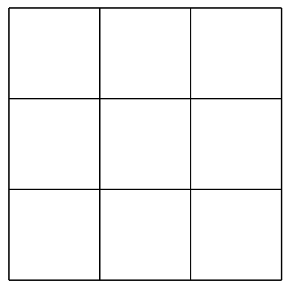
MthT 491 Toothpick Squares and Triangles

Motivated by Mark Driscoll, **Fostering Algebraic Thinking: A Guide for Teachers Grade 6-10** [**Driscoll**].

We use toothpicks to construct *toothpick squares*. The square with side length 3 [toothpicks] is shown below:



We think of the big square as *simple graph* in the sense of graph theory. For the simplest introduction to a simple graph, I quote from

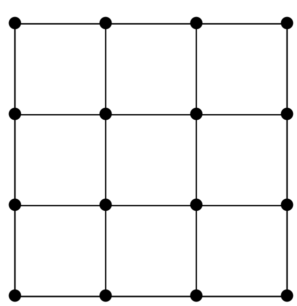
`http://mathworld.wolfram.com/Graph.html`:

> In a mathematician's terminology, a graph is a collection of points and lines connecting some (possibly empty) subset of them. The points of a graph are most commonly known as graph vertices, but may also be called "nodes" or simply "points." Similarly, the lines connecting the vertices of a graph are most commonly known as graph edges, but may also be called "arcs" or "lines."
>
> ... [G]raphs in which at most one edge (i.e., one edge or no edge) may connect any two vertices are said to be *simple graphs*.

In our example, the collection of points (nodes, vertices) are the endpoints of the toothpicks and the toothpicks are the *graph edges*.

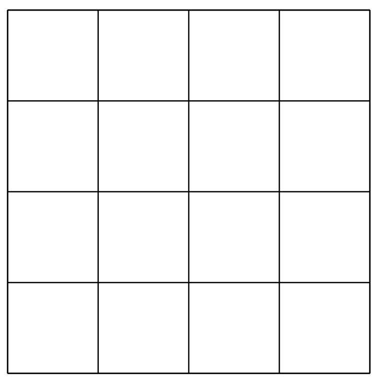
The degree of a graph vertex P of a graph G is the number of graph edges which touch P.

Let's consider the square of side length 3 as a simple graph.



1. How many graph edges (toothpicks) are there? $(2 \cdot 3\,(3+1) = 24?)$
2. On the graph, label the degree of each vertex.
3. How many *vertices* are there in the graph of the square with $n = 4$?
4. How many vertices have degree 1?
5. How many vertices have degree 2?
6. How many vertices have degree 3?
7. How many vertices have degree 4?
8. What is the sum of the the degrees of all nodes?

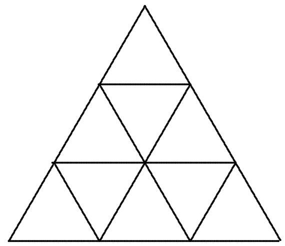
Now change the the side length of the square to $n = 4$ [toothpicks] and answer the same questions. You may wish to draw the graph.



For the toothpick square with side length n, try to find rules for the number of vertices of degree 2, 3, and 4.

Toothpick Equilateral Triangles

Next we use toothpicks to construct *toothpick equilateral triangles*. The triangle with side length 3 [toothpicks] is shown below:



Think of the big triangle square as *simple graph* in the sense of graph theory.

The graph vertices (nodes, points) are the endpoints of the toothpicks and the toothpicks are the *graph edges*.

Let's consider the triangle of side length 3 [toothpicks] as a simple graph.

1. How many graph edges (toothpicks) are there?
2. On the graph, label the degree of each vertex.
3. How many *vertices* are there?
4. How many vertices have degree 1?
5. How many vertices have degree 2?
6. How many vertices have degree 3?
7. How many vertices have degree 4?
8. How many vertices have degree 5?
9. How many vertices have degree 6?
10. How many vertices have degree 7?

Now construct the *toothpick equilateral triangle* with side length $n = 4$ [toothpicks]. Find the number of vertices of degree $1, 2, \ldots, 7$.

The **Assignment** is on the next page.

The Assignment

For the toothpick equilateral triangle with side length n, try to find rules for the number of vertices of degree 2, 3, 4, 5, and 6. Carefully explain how you find the rule(s).

Cooperation: You are encouraged to work together. Your writeup should acknowledge the contributions of others in your group. Your writeup should reflect your understanding of the patterns, methods, and rules that you find.

Due Dates: Give a progress report via e-mail or in class Tuesday, September 16, 2003. Turn in your final paper Tuesday, September 23, 2003. Type your paper. Use only complete sentences. You are probably better at graphics than I am, but hand drawn figures are acceptable.

N.B. To simplify a counting rule related to a triangle, you may find the following formula useful:

$$1 + 2 + \cdots + n = \frac{1}{2}\left(n\left(n+1\right)\right).$$

This is interesting in itself. One proof is given below.

$$\begin{aligned}
\sum_{k=1}^{n} k &= 1 + 2 + \ldots + n \\
&= n + (n-1) + \ldots + 1 \\
&= \frac{1}{2}\left(\left(1 + 2 + \ldots + n\right) + \left(n + (n-1) + \ldots + 1\right)\right) \\
&= \frac{1}{2}\left(\left(1+n\right) + \left(2 + (n-1)\right) + \ldots + \left(n+1\right)\right) \\
&= \frac{1}{2}\left(n\left(n+1\right)\right).
\end{aligned}$$