Seminar on Characteristic classes

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December 23, 2013

Contents

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Seminars were organized around Milnor and Stasheff's *Characteristic Classes.*

1 Seminars

1.1 Background and Stiefel-Whitney classes

We will begin by stating the axioms of the Stiefel-Whitney class, and then proceeding to build up all the knowledge required to understand them.

Definition 1.1. The Stiefel-Whitney characteristic class is defined by the following axioms.

A1. For each $i \in \mathbb{Z}_{\geqslant 0}$, there is an element $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$, called the ith *Stiefel-Whitney class*, with

$$\begin{aligned} w_0(\xi) &= 1 \\ w_i(\xi) &= 0 \qquad \text{if } i > n \text{ and } \xi \text{ is an } \mathbb{R}^n\text{-bundle} \end{aligned}$$

A2. If there exists a bundle map from ξ to η with $f : B(\xi) \to B(\eta)$, then $w_i(\xi) = f^* w_i(\eta)$

A3. If ξ and η are vector bundles over the same space, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta)$$

A4. For the canonical line bundle over $\mathbb{P}^1$, $w_1(\gamma_1^1) \neq 0$.

Definition 1.2. A *real vector bundle* ξ over B consists of the following:

1. A topological space $E = E(\xi)$ called the *total space*
2. A continuous map $\pi : E \to B$ called the *projection map*
3. For each $b \in B$, fhe fiber $\pi^{-1}(b) = F_b$ is a vector space

The condition of *local triviality* must also be satisfied: for each $b \in B$, there is a neighborhood $U \ni b$ and a homeomorphism h and an isomorphism φ, given by

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U) \qquad\qquad \begin{array}{rccl} \varphi : & \mathbb{R}^n & \to & \pi^{-1}(b) \quad \text{for all } b \in U \\ & x & \mapsto & h(b,x) \end{array}$$

The pair (U, h) is termed a *local coordinate system* for ξ.

Definition 1.3. A vector bundle ξ is termed an *n-plane bundle* or an *$\mathbb{R}^n$-bundle* if $\pi^{-1}(b) \cong \mathbb{R}^n$ for all $b \in B$.

Note that the fiber function F_b must be a locally constant function of b.

Definition 1.4. Let ξ be a vector bundle with $E \xrightarrow{\pi} B$, and let B_1 be an arbitrary topological space such that $f : B_1 \to B$ is continuous. Define the *induced bundle* or *pullback bundle* $f^*\xi$ over B_1 to consist of

1. the total space $E_1 = \{(b_1, e) \in B_1 \times E \ : \ f(b_1) = \pi(e)\}$
2. the projection map $\pi_1 : E_1 \to B_1$ given by $\pi_1(b_1, e) = b_1$
3. for each $b_1 \in B_1$, the fiber F_{b_1} is isomorphic by $\hat{f}$ to $F_{f(b_1)}$

Local triviality is satisfied by: for each $b_1 \in B$, there is a neighborhood $U_1 = f^{-1}(U)$, for U the neighborhood of $f(b_1)$, and a homeomorphism

$$\begin{array}{rccl} h_1 : & U_1 \times \mathbb{R}^n & \to & \pi_1^{-1}(U_1) \\ & (b,x) & \mapsto & (b_1, h(f(b_1), x)) \end{array}$$

The local coordinate system for $f^*\xi$ is (U_1, h_1). This information is contained within the following commutative diagram:

$$\begin{tikzcd}
E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\
B_1 \arrow[r, "f"'] & B
\end{tikzcd}$$

It may be demonstrated that if ξ is a smooth bundle and f is smooth, then $f^*\xi$ is a smooth bundle.

Definition 1.5. Let ξ, η be vector bundles of the same rank. A *bundle map* $f : \xi \to \eta$ is a map in the category of vector bundles, as well as a continuous function $f : E(\xi) \to E(\eta)$ such that $F_b(\xi) \cong F_{b'}(\eta)$ as vector spaces, via g.

So g carries each η-fiber isomorphically over to some ξ-fiber. Further, the bundle map induces a function of base spaces $\bar{f} : B(\xi) \to B(\eta)$.

Lemma 1.6. Let ξ, η be vector bundles and $g : \eta \to \xi$ a bundle map. Then $\eta \cong \bar{g}^*\xi$.

Definition 1.7. Let ξ_1, ξ_2 be vector bundles over B. The *Whitney sum* of ξ_1 and ξ_2 is the induced bundle $d^*(\xi_1 \times \xi_2)$, and is denoted by $\xi_1 \oplus \xi_2$, where $d : B \to B \times B$ is given by $b \mapsto (b, b)$. The isomorphism $F_b(\xi_2 \oplus \xi_2) \cong F_b(\xi_1) \oplus F_b(\xi_2)$ is canonical.

Definition 1.8. The *canonical line bundle* over $\mathbb{R}P^n$, denoted by γ_n^1, consists of

1. the total space $E = \{(\pm x, v) \ : \ v = \lambda x, \lambda \in \mathbb{R}\} \subset \mathbb{R}P^1 \times \mathbb{R}^{n+1}$
2. the projection map $\pi : E \to \mathbb{R}P^n$ given by $\pi(\pm x, v) = \pm x$
3. for each $x \in \mathbb{R}P^n$, the fiber F_x is associated to the line through x and $-x$ in $\mathbb{R}^{n+1}$

Local triviality is satisfied by choosing for each $x \in \mathbb{R}P^n$ a neighborhood $U \subset \mathbb{S}^n$ small enough to contain no pair of antipodal points. Let U_1 be the image of U in $\mathbb{R}P^n$. Then the map

$$\begin{array}{rccc} h : & U_1 \times \mathbb{R} & \to & \pi^{-1}(U_1) \\ & (\pm x, t) & \mapsto & (\pm x, tx) \end{array}$$

is a homeomorphism, so (U_1, h) is a local coordinate system.

Now we have all the basic definitions to understand the Stiefel-Whitney class. Let us do some simple examples.

Example 1.9. Calculate $w(\gamma_n^1)$.

1.2 Grassmannians

Definition 2.1. The *Grassmannian manifold* $Gr(n,k) = G_n(\mathbb{R}^{n+k}) = Gr(n,V)$ is the set of all n-dimensional vector subspaces of $\mathbb{R}^n$ (or $\mathbb{R}^{n+k}$, or V, respectively).

Example 2.2. The real projective space may be expressed as $Gr(1,\mathbb{R}^k) = \mathbb{R}P^{k-1}$.

Proposition 2.3. $Gr(n,\mathbb{R}^k)$ is a compact manifold.

<u>*Proof:*</u> (sketch) Let $V, W \in Gr(n,\mathbb{R}^k)$. If W is "close" to V (i.e. not orthogonal), then W is the graph of a linear transformation $f_W : V(\cong \mathbb{R}^n) \to V^\perp(\cong R^{k-n})$. Since f_W is linear, there is an $n \times (k-n)$ matrix representing f_W. By injectively mapping subspaces to matrices, we get a natral isomorphism with $\mathbb{R}^{n(k-n)}$.

Construction and injectivity of the homeomorphism $\{W\} \to \mathrm{Hom}(V,V^\perp)$, as well as compactness, are left to the reader. Compactness may be proved by constructing a diffeomorphism

$$Gr(n,\mathbb{R}^k) \cong O(k)\Big/[O(n) \times O(k-n)]$$

$\blacksquare$

Definition 2.4. The *tautological vector bundle* (or *canonical vector bundle*) $\gamma^n(\mathbb{R}^k)$ over $Gr(n,\mathbb{R}^k)$ consists of

1. the total space $E = \{(n\text{-dim. vec. subsp. of } \mathbb{R}^k, \text{ vector in that subsp.})\}$
2. the projection map $\pi : E \to Gr(n,\mathbb{R}^k)$ defined by $\pi(X,x) = X$
3. fibers F_X with vec. sp. structure $t_1(X,x_1) + t_2(X,x_2) = (X, t_1x_1 + t_2, x_2)$

The triviality condition is satisfied by taking each $X \in Gr(n,\mathbb{R}^k)$, and letting $U = \{Y \ : \ Y \cap X^\perp = 0\} \subset Gr(n,\mathbb{R}^k)$, and defining a homeomorphism h by:

$$\begin{array}{rccll} h: & U \times X & \to & \pi^{-1}(U) & \\ & (Y,x) & \mapsto & (Y,y) & \qquad \text{such that } \mathrm{proj}_X(y) = x \end{array}$$

Observe the following inclusion, which holds as $\mathbb{R}^k \subset \mathbb{R}^{k+1}$:

$$Gr(n,\mathbb{R}^k) \hookrightarrow Gr(n,\mathbb{R}^{k+1})$$

By taking the limit of these inclusions, as $k \to \infty$, we get a new object.

Definition 2.5. The *infinite Grassmannian manifold* $Gr(n) = GR_n = Gr(n,\mathbb{R}^\infty)$ is the set of all n-dimensional vector subspaces of $\mathbb{R}^\infty$.

As a set, $Gr(n) = \bigcup\limits_{k \geqslant n} Gr(n,\mathbb{R}^k)$, which is a direct limit. Let us consider the construction of direct limits.

For a system $(Gr(n,\mathbb{R}^k), \iota_{k\ell})$ as below for all $\ell \geqslant k$, the direct limit of the system is the object $Gr(n)$, equipped with maps ι_k, ι_ℓ such that $\iota_k = \iota_\ell \circ \iota_{k\ell}$. Further, if A is any other object with maps φ_k, φ_ℓ as in the diagram below, then there exists a unique map φ such that $\varphi_\ell = \varphi \circ \iota_\ell$.

$$\begin{tikzcd}
Gr(n,\mathbb{R}^k) \arrow[dr, "\iota_k"] \arrow[dd, "\iota_{k\ell}"'] \arrow[drr, bend left, "\varphi_k"] & & \\
& Gr(n) \arrow[r, dashed, "\varphi"] & A \\
Gr(n,\mathbb{R}^\ell) \arrow[ur, "\iota_\ell"'] \arrow[urr, bend right, "\varphi_\ell"'] & &
\end{tikzcd}$$

What is the topology on $Gr(n)$? We take it to be the largest possible topology:

$$U \subset Gr(n) \text{ is open/closed} \quad \Longleftrightarrow \quad (U \cap Gr(n, \mathbb{R}^k)) \subset Gr(n,k) \text{ is open/closed for all } k \geqslant n$$

Now let's consider what the tautological vector bundle over $Gr(n)$ looks like.

Definition 2.6. The *universal vector bundle* γ^n over $Gr(n)$ has the exact same structure as $\gamma^n(\mathbb{R}^k)$, except $\mathbb{R}^\infty$ is used instead of $\mathbb{R}^k$.

Definition 2.7. A *paracompact space* is a Hausdorff space such that every open cover has a locally finite open refinement.

This means that every point has an open neighborhood that is in finitely many elements of the refined cover.

Example 2.8. Every metric space is paracompact.

Example 2.9. Every manifold (space that is locally Euclidean) that is Hausdorff and has a countable topological basis in paracompact.

Now that we have paracompactness, we can understand the proofs of the next two theorems (although they are presented without proofs).

Theorem 2.10. For an $\mathbb{R}^n$-bundle ξ over B paracompact, there exists a bundle map $\xi \to \gamma^n$.

Theorem 2.11. Any two bundle maps f, g from an $\mathbb{R}^n$ bundle ξ to γ^n are bundle-homotopic.

Definition 2.12. Two bundle maps $f, g : \xi \to \eta$ are *bundle-homotopic* if there exists a continuous map

$$h : [0,1] \times E(\xi) \to E(\eta)$$

that is continous in both variables, with $h_0 = f$, $h_1 = g$. Moreover, $h_t : E(\xi) \to E(\eta)$ is a bundle map for all $t \in [0,1]$.

These two theorems imply the following:

Corollary 2.13. Any $\mathbb{R}^n$ bundle ξ over B paracompact determines a homotopy class of maps $\bar{f}_\xi : B \to Gr(n)$.

<u>*Proof:*</u> Given any bundle map $f : \xi \to \gamma^n$, let $\bar{f}$ be the induced map of base spaces. To see that any other bundle map $g : \xi \to \gamma^n$ will have an induced map homotopic to $\bar{f}_\xi$, consider the folowing commutative diagram.

$$\begin{tikzcd}
E(\xi) \arrow[r, "{f,g}"] \arrow[d, "\pi_\xi"'] & E(\gamma^n) \arrow[d, "\pi_\gamma"] \\
B \arrow[r, "{\bar{f},\bar{g}}"'] & Gr(n)
\end{tikzcd}$$

Given a homotopy h between f and g, we may construct a homotopy $\bar{h}$ by

$$\begin{array}{rrcll}
\bar{h}: & [0,1] \times B & \to & Gr(n) & \\
 & \bar{h}_t(b) & = & (\pi_\gamma \circ h_t)(F_b) & \forall\, b \in B,\ t \in [0,1]
\end{array}$$

Hence $\bar{f}$ is homotopic to $\bar{g}$, and they are in the same class. ■

Definition 2.14. Let A be a coefficient group or ring, and choose $c \in H^i(Gr(n); A)$ and a vector bundle ξ. Then $c(\xi) = \bar{f}_\xi^* c \in H^i(B, A)$ is termed the *characteristic cohomolgy class* (or simply *characteristic class*) of ξ determined by c.

1.3 The CW-structure of $Gr(n)$

We begin with a motivating theorem.

Theorem 3.1. The space $Gr(n, \mathbb{R}^k)$ is a CW-complex with Schubert cells $e(\sigma)$, for k finite and in the direct limit $k \to \infty$.

To understand this theorem, we need to define what Schubert cells are, and that they actually are cells. We begin by recalling the inclusion of spaces considered proviously:

$$\begin{array}{c} \mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^n \subset \cdots \subset \mathbb{R}^k \\ \| \\ \{(v_1, \dots, v_n, 0, \dots, 0) \ : \ v_i \in \mathbb{R}\} \end{array}$$

For each n-dimensional vector subspace $V \subset \mathbb{R}^k$, there exists a sequence of integers

$$0 = \dim(V \cap \mathbb{R}^0) \leqslant \dim(V \cap \mathbb{R}^1) \leqslant \cdots \leqslant \dim(V \cap \mathbb{R}^k) = n$$

with the property that $\dim(V \cap \mathbb{R}^{i+1}) - \dim(V \cap \mathbb{R}^i) \leqslant 1$ for all i. This property can be seen by considering the short exact sequence

$$0 \longrightarrow V \cap \mathbb{R}^i \xrightarrow{\iota} V \cap \mathbb{R}^{i+1} \xrightarrow{\pi_{i+1}} X \longrightarrow 0$$

Here ι is the standard inclusion map, and π_{i+1} is the projection of the $(i+1)$th coordinate of the preceding space onto $\mathbb{R}$. We note the following:

$$\begin{aligned} \operatorname{Im}(\iota) = V \cap \mathbb{R}^i \qquad & \implies \qquad X \cong (V \cap \mathbb{R}^{i+1})/(V \cap \mathbb{R}^i) \\ \ker(\pi_{i+1}) = V \cap \mathbb{R}^i \qquad & \implies \dim(X) = \dim(V \cap \mathbb{R}^{i+1}) - \dim(V \cap \mathbb{R}^i) \\ & \qquad 1 \text{ or } 0 = \dim(V \cap \mathbb{R}^{i+1}) - \dim(V \cap \mathbb{R}^i) \end{aligned}$$

In the case that $\dim(V \cap \mathbb{R}^i) = \dim(V \cap \mathbb{R}^{i+1})$, then $X = 0$, and π_{i+1} is the zero map. Otherwise, it must be that $X = \mathbb{R}$, and so $\dim(V \cap \mathbb{R}^{i+1}) - \dim(V \cap \mathbb{R}^i) = 1$.

Let us now introduce some necessary definitions.

Definition 3.2. An *n-frame* in $\mathbb{R}^k$ is a linearly independent set $S \subset \mathbb{R}^k$ with $|S| = n$.

Definition 3.3. The *Stiefel manifold* $V_n(\mathbb{R}^k) \subset (\mathbb{R}^k)^{\times n}$ is the collection of all n-frames in $\mathbb{R}^k$. The manifold $V_n^o(\mathbb{R}^k)$ is the collection of all orthonormal frames in $\mathbb{R}^k$.

Definition 3.4. The *Schubert symbol* $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{Z}_+^n$ is a sequence of positive integers that satisfies

$$1 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n \leqslant k$$

The *Schubert cell* is defined to be the set

$$e(\sigma) = \{V \in Gr(n, \mathbb{R}^k) \ : \ \dim(V \cap \mathbb{R}^{\sigma_i}) - \dim(V \cap \mathbb{R}^{\sigma_i - 1}) = 1\}$$

Note that for each i, $\dim(V \cap \mathbb{R}^i)$ is the same for all $V \in e(\sigma)$. Moreover, we note that each $V \in Gr(n, \mathbb{R}^k)$ lives in exactly one of the $\binom{k}{n}$ sets $e(\sigma)$.

Definition 3.5. Let H^n denote the open half-space in $\mathbb{R}^k$ given by

$$H^n = \{(x_1, \dots, x_n, 0, \dots, 0) \in \mathbb{R}^k \ : \ x_n > 0\}$$

Note that for $V \in Gr(n)$, $V \in e(\sigma)$ iff there exists a basis $\{v_1, \dots, v_n\}$ of V with $v_i \in H^{\sigma_i}$ for all i.

Lemma 3.6. Each n-plane $V \in e(\sigma)$ has a unique orthonormal basis $(v_1, \ldots, v_n) \in H^{\sigma_1} \times \cdots \times H^{\sigma_n}$.

Proof: The proof works by induction on n. For the base case $n = 1$, we have $v_1 \in V \cap \mathbb{R}^{\sigma_1}$. This is a 1-dimensional space, and the vector must be normal and have positive entries. This completely defines the vector v_1.

For $v_i \in V \cap \mathbb{R}^{\sigma_i}$, we have that the space is i-dimensional, and all the vectors v_j for $1 \leqslant j < i$ have been defined as desired. As v_i is orthogonal to all v_j for $1 \leqslant j < i$, and it is normal with positive entries, we have a completely defined vector. ■

We now will show that the Schubert cells are actually cells.

Definition 3.7. Define the following objects:

$$\begin{aligned}
\overline{e}(\sigma) &= \mathrm{cl}(e(\sigma)) \\
e'(\sigma) &= V_n^o(\mathbb{R}^k) \cap (H^{\sigma_1} \times \cdots \times H^{\sigma_n}) \\
\overline{e}'(\sigma) &= V_n^o(\mathbb{R}^k) \cap (\mathrm{cl}(H^{\sigma_1}) \times \cdots \times \mathrm{cl}(H^{\sigma_n}))
\end{aligned}$$

The object $\overline{e}(\sigma)$ is called the *Schubert variety.* The object $e'(\sigma)$ consists of orthonormal n-frames in $\mathbb{R}^k$ with the ith coordinate in H^{σ_i} for all i.

Lemma 3.8.
1. $\overline{e}(\sigma)$ is a closed cell of dimension $\sum_{i=1}^n (\sigma_i - i)$ with $\mathrm{int}(\overline{e}'(\sigma)) = e'(\sigma)$
2. There exists a homeomorphism

$$\begin{array}{rccc}
q: & e'(\sigma) & \to & e(\sigma) \\
 & \overline{e}'(\sigma) & \to & \overline{e}(\sigma)
\end{array}$$

Proof: Only a sketch of the proof is provided. This is done by induction on n. For $n = 1$, we observe that

$$\begin{aligned}
\overline{e}'(\sigma_1) &= \left\{ x_1 = (x_{11}, x_{12}, \ldots, x_{1\sigma_1}, 0, \ldots,) \ : \ \sum x_{1i}^2 = 1, x_{1\sigma_1} > 0 \right\} \\
&= (\text{closed hemisphere of dimension } \sigma_i - 1) \\
&\cong D^{\sigma_i - 1} \\
&= (\text{cell of dimension } \sigma_i - 1)
\end{aligned}$$

For the inductive case, let $T(u, v)$ be the unique map that rotates $\mathbb{R}^k$ so that u goes to v, and everything orthogonal to both u and v stays fixed. Let

$$b_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in H^{\sigma_i} \subset \mathbb{R}^k$$

where the 1 is in the ith position. For any n-fram $(x_1, \ldots, x_n)$, define the map

$$T = T(b_n, x_n) \circ T(b_{n-1}, x_{n-1}) \circ \cdots \circ T(b_1, x_1)$$

So $b_i \mapsto x_i$ by T for all $i = 1, \ldots, n$. Now, for some $\sigma_{i+1} > \sigma_i$, we let

$$\begin{aligned}
D &= \{u \in \mathrm{cl}(H^{\sigma_{i+1}}) \ : \ b_i \cdot u = 0 \ \forall \ i\} \\
&= (\text{the closed hemisphere of dimension } \sigma_{n+1} - n - 1) \\
&\cong D^{\sigma_{n+1} - (n+1)} \\
&= (\text{cell of dimension } \sigma_{n+1} - (n+1))
\end{aligned}$$

Now we define a homeomrphism

$$\begin{array}{rccc}
q: & \overline{e}'(\sigma_1, \ldots, \sigma_n) \times D & \to & \overline{e}'(\sigma_1, \ldots, \sigma_{n+1}) \\
 & ((x_1, \ldots, x_n), u) & \mapsto & (x_1, \ldots, x_n, Tu)
\end{array}$$

This maps also works for $e'(\sigma) \to e(\sigma)$. ■

With the developed tools, we may now prove Theorem 3.1. Let us restate it:

Theorem 3.1. The space $Gr(n, \mathbb{R}^k)$ is a CW-complex with Schubert cells $e(\sigma)$, for k finite and in the direct limit $k \to \infty$.

Proof: It must be shown that the boundary of a cell $e(\sigma)$ lies in a cell $e(\tau)$ of a lower dimension. The boundary of $e(\sigma)$ is $\overline{e}(\sigma) - e(\sigma) = q(\overline{e}'(\sigma)) - e(\sigma)$ by the previous theorem.

Then note that an n-plane V in the boundary has an orthonormal basis $\{v_1, \ldots, v_n\}$ with $v_i \in \mathbb{R}^{\sigma_i}$. As $V \notin e(\sigma)$, there is at least one $v_i \in \mathbb{R}^{\sigma_i - 1}$ (all the other $v_i \in \mathbb{R}^{\sigma_i}$). So then the Schubert symbol $(\tau_1, \ldots, \tau_n)$ associated with V has $\tau_i < \sigma_i$, so $\dim(\tau) < \dim(\sigma)$.

Hence $Gr(n, \mathbb{R}^k)$ is a CW-complex. Similarly, $Gr(n)$ is a CW-complex, as $V \in Gr(n, \mathbb{R}^k)$ for some finite k. In addition, the topology on $Gr(n)$ is the direct limit of the topology on $Gr(n, \mathbb{R}^k)$. ■

To conclude, we will introduce orientation.

Definition 3.7. An *orientation* of a real vector space V is an equivalence class of bases. Two ordered bases are equivalent iff the change of basis matrix has positive determinant.

There are clearly only two such equivalence classes.

Remark 3.8. A choice of orientation for V corresponds to a choice of one of two possible generators of the reduced homology $H_n(V, V_0; \mathbb{Z})$, where V_0 is the set of non-zero vectors of V.

In the next lecture, we will discuss the Chern class, which deals with bundles that have a natural orientation.

1.4 Chern classes: Part 1

Last lecture we ended on orientation. Let's deifne orientation on a fiber bundle.

Definition 4.1. Let ξ be a $\mathbb{R}^n$-bundle. A *pre-orientation* on ξ is a choice of orientation on each fiber F_b. A pre-orientation is an *orientation* if for each $b \in B$ there exists an open neighborhood $U \ni b$ with trivialization $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the restriction $h|_{F_b} : F_b \to b \times \mathbb{R}^n$ preserves orientation.

The space $\mathbb{R}^n$ in the image of h is given the orientation induced by the standard basis.

We now introduce complex bundles and bundles related to them, which will be used in the definition of the Chern classes.

Definition 4.2. A *complex vector bundle* ω of complex dimension n (a $\mathbb{C}^n$-bundle) over B consists of

1. the total space E
2. the projection map $\pi : E \to B$
3. for each $b \in B$, fhe fiber $\pi^{-1}(b) = F_b$ has a complex vector space structure

Local triviality is satisfied by stating that for all $b \in B$, there exists an open neighberhood $U \ni b$ in B such that $\pi^{-1}(U) \cong U \times \mathbb{C}^n$, where $\cong$ is homeomorphism, and $\pi^{-1}(b)$ is mapped complex linearly onto $b \times \mathbb{C}^n$.

Definition 4.3. Given a $\mathbb{C}^n$-bundle ω, the *underlying* $\mathbb{R}^{2n}$*-bundle* $\omega_{\mathbb{R}}$ has the structure of ω, except that each fiber has the structure of a real vector space, and $\pi^{-1}(U) \cong U \times \mathbb{R}^{2n}$.

Now we are ready to introducethe Chern class. The Euler class is used in the definition, but the exposition of the Euler class is left for a later time.

Definition 4.4. Let ω be a $\mathbb{C}^n$-bundle. The Chern classes $c_i(\omega) \in H^{2i}(B;\mathbb{Z})$ are defined by induction on the complex dimension n of ω as follows:

· $i < n$: $c_i(\omega) = (\pi_0^*)^{-1} c_i(\omega_0)$
· $i = n$: $c_i(\omega) = e(\omega_{\mathbb{R}})$
· $i > n$: $c_i(\omega) = 0$

The formal sum $c(\omega) = 1 + c_1(\omega) + \cdots + c_n(\omega)$ is termed the *total Chern class*.

Remark 4.5. The bundle ω_0 indicated above is the bundle that has E_0, the set of all non-zero vectors in E, as its base space. The relation between this bundle and ω is demonstrated in the following diagram:

$$\begin{tikzcd}
 & E \arrow[dd, "\pi"'] & & \omega_0 & \\
\omega & & E_0 \arrow[ul, "\iota"'] \arrow[dl, "\pi \circ \iota"] & & X \arrow[ll, "\pi_0"'] \\
 & B & & &
\end{tikzcd}$$

$$\begin{aligned}
E_0 &= \{(p, v_p) \ : \ p \in B, v_p \in F_b\} \\
X &= \{(q, v_q) \ : \ q = (p, v_p) \in E_0, v_q \in F_q/\mathbb{C}v_p\}
\end{aligned}$$

This also shows where the map π_0 is coming from. The induced map on cohomology, π_0^*, is used in the following theorem.

Theorem 4.6. [Gysin]
Let ξ be an oriented $\mathbb{R}^n$-bundle. Then there exists an exact sequence, with coefficients over $\mathbb{Z}$, given by:

$$\cdots \longrightarrow H^i(B) \xrightarrow{\smile e(\xi)} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \longrightarrow H^{i+1}(B) \xrightarrow{\smile e(\xi)} \cdots$$

The proof of this theorem is not presented here. In may be found in Milnor and Stasheff, section 12.

Remark 4.7. It still remains to show that the inverse of π_0^* is well-defined. We do this by showing that $H^i(B) \cong H^{i+1}(B) \cong 0$ for $-2n \leqslant i < 0$.

Remark 4.8. Some facts about the Chern classes:

· If $g : \omega \to \omega'$ is a bundle map, then $c(\omega) = f^* c(\omega')$ for $f : B \to B'$ induced by g

· If ε^n is the trivial $\mathbb{C}^n$ bundle over B, then $c(\omega \oplus \varepsilon^n) = c(\omega)$

Example 4.9. Consider $\mathbb{C}P^n = Gr(1, \mathbb{C}^{n+1})$ = the base space of the complex line bundle γ^1, a 1-dimensional bundle. Since it is one dimensional, $c_1(\gamma^1) = e(\gamma^1)$. This allows us to write the Gysin sequence as:

$$\cdots \longrightarrow H^i(E_0) \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\smile c_1(\gamma^1)} H^{i+2}(\mathbb{C}P^n) \xrightarrow{\pi_0^*} H^{i+2}(E_0) \longrightarrow H^{i+1}(\mathbb{C}P^n) \longrightarrow \cdots$$

Consider the space E_0, which may be described as:

$$E_0 = \{(\text{line through origin in } \mathbb{C}^{n+1}, \text{ non-zero vector in that line})\} \cong \mathbb{C}^{n+1} \setminus \{0\}$$

As $\mathbb{C}^{n+1}$ looks like $\mathbb{R}^{2n+2}$, it follows that $\mathbb{C}^{n+1} \setminus \{0\}$ has the same homotopy type as $\mathbb{S}^{2n+1}$. For this sphere, we know that

$$H^i(\mathbb{S}^i; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2n+1 \\ 0 & \text{else} \end{cases}$$

This allows us to simplify the Gysin sequence above, as

$$0 \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\smile c_1(\gamma^1)} H^{i+2}(\mathbb{C}P^n) \longrightarrow 0$$

for all $0 \leqslant i \leqslant 2n-2$, so the two indicated groups are isomorphic for all such i. Since $\mathbb{C}P^n$ is compact, connected and orientable, its zeroth cohomology class is $\mathbb{Z}$, so

$$\mathbb{Z} \cong H^0(\mathbb{C}P^n) \cong H^2(\mathbb{C}P^n) \cong \cdots \cong H^{2n}(\mathbb{C}P^n)$$

From the cup product map, we have that H^{2i} is generated by $c_1(\gamma^1)^i$. Further, by adjusting the indeces of the Gysin sequence, we get a similar equivalence for the odd groups:

$$0 \cong H^{-1}(\mathbb{C}P^n) \cong H^1(\mathbb{C}P^n) \cong \cdots \cong H^{2n-1}(\mathbb{C}P^n)$$

In the next lecture, we will discuss some interesting properties of the Chern classes.

1.5 Chern classes: Part 2

This lecture will be concerned with proving the product theorem, namely, that $c(\omega \oplus \phi) = c(\omega)c(\phi)$ for ω, ϕ complex bundles over the same B paracompact. Before we can prove that, we need some auxiliary statements. Compare the first with Theorem 2.10.

Lemma 5.1. Let ω be a $\mathbb{C}^n$-bundle over B paracompact. Then there exists a bundle map $\omega \to \gamma^n$ over $Gr(n, \mathbb{C}^\infty) = Gr(n)$.

The proof to this is much the same as the proof to 2.10, and so is omitted here.

Lemma 5.2. The cohomology ring $H^\bullet(Gr(n); \mathbb{Z})$ is a polynomial ring over $\mathbb{Z}$ generated by $c_1(\gamma^n), \dots, c_n(\gamma^n)$.

The proof to this is quite long. The interested reader is reffered to Theorem 14.5 in [2].

Lemma 5.3. Let ω over B be a complex bundle and ε the trivial $\mathbb{C}^n$-bundle over B. Then $c(\omega \oplus \varepsilon) = c(\omega)$.

The proof to this is not as long, but is still omitted. We now move on to proving a statement.

Lemma 5.4. There exists a unique polynomial $p_{m,n} \in \mathbb{Z}[c_1, \dots, c_m, c_1', \dots, c_n']$ so that for every $\mathbb{C}^m$-bundle ω and $\mathbb{C}^n$-bundle ϕ, both over B paracompact:

$$c(\omega \oplus \phi) = p_{m,n}(c_1(\omega), \dots, c_m(\omega), c_1(\phi), \dots, c_n(\phi))$$

Proof: Recall that we have the canonical vector bundles γ^m, γ^n over $Gr(m)$ and $Gr(n)$, respectively. So let a new base space be $Gr(m) \times Gr(n)$. We get new bundles from maps induced by the two projections to each factor of this space:

$$\begin{array}{rcl} \pi_1 : Gr(m) \times Gr(n) \to Gr(m) & \quad\text{induces}\quad & \pi_1^* : \gamma^m \to \gamma_1^m \\ \pi_2 : Gr(m) \times Gr(n) \to Gr(n) & \quad\text{induces}\quad & \pi_2^* : \gamma^n \to \gamma_2^n \end{array}$$

Lemma 5.1 guarantees the existence of bundle maps f_1 and f_2 as below. We will first prove this theorem for bundles γ^m and γ^n, and then extend the result.

$$\begin{array}{ccccc} \omega & \xrightarrow{f_1} & \gamma^m & \xrightarrow{\pi_1^*} & \gamma_1^m \\ \phi & \xrightarrow[f_2]{} & \gamma^n & \xrightarrow[\pi_2^*]{} & \gamma_2^n \end{array}$$

So γ_1^m and γ_2^n are both bundles over $Gr(m) \times Gr(n)$. Hence the Whitney sum bundle $\gamma_1^m \oplus \gamma_1^n$ is isomorphic to the bundle $\gamma^m \times \gamma^n$, as the fibers are $F^m \times F^n \cong F^m \oplus F^n$.

Consider the cohomology cross product (see Definition 3.2 in Section 2.3) given by

$$\begin{array}{rrcl} \times : & H^k(Gr(m); \mathbb{Z}) \otimes_{\mathbb{Z}} H^\ell(Gr(n); \mathbb{Z}) & \to & H^{k+\ell}(Gr(m) \times Gr(n); \mathbb{Z}) \\ & a \otimes_{\mathbb{Z}} b & \mapsto & \pi_1^*(a) \smile \pi_2^*(b) \end{array}$$

The fact that $\times$ is actually an isomorphism follows from the Künneth fomula (Theorem 3.3 in Section 2.3). By Lemma 5.2, the space $H^{k+\ell}(Gr(m) \times Gr(n); \mathbb{Z})$ is generated by

$$\{\pi_1^* c_i(\gamma^m) = c_i(\gamma_1^m) \ : \ 1 \leqslant i \leqslant m\} \quad \cup \quad \{\pi_2^* c_j(\gamma^n) = c_j(\gamma_2^n) \ : \ 1 \leqslant j \leqslant n\}$$

Again using Lemma 5.2, we have that the total Chern class of $\gamma_1^m \oplus \gamma_2^n$ is given by the unique polynomial

$$c(\gamma_1^m \oplus \gamma_2^n) \in \mathbb{Z}[c_1(\gamma_1^m), \dots, c_m(\gamma_1^m), c_1(\gamma_2^n), \dots, c_n(\gamma_2^n)]$$

Now let us extend this result. Let ω be a $\mathbb{C}^m$-bundle and ϕ a $\mathbb{C}^n$-bundle, both over B paracompact. By Theorem 5.1, there exist maps $f : B \to Gr(m)$ and $g : B \to Gr(n)$ that induce bundle maps f^* and g^*, with $f^*(\gamma^m) \cong \omega$ and $g^*(\gamma^n) \cong \phi$. Define a map

$$\begin{array}{rrcl} h: & B & \to & Gr(m) \times Gr(n) \\ & b & \mapsto & (f(b), g(b)) \end{array}$$

This gives a commutative diagram:

$$\begin{tikzcd}
 & B \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
Gr(m) & & Gr(n) \\
 & Gr(m) \times Gr(n) \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Therefore h induces a bundle map h^* with $h^*(\gamma_1^m) \cong \omega$ and $h^*(\gamma_2^n) \cong \phi$. By the axioms of Chern classes (and characteristic classes in general), the total class of $\omega \oplus \phi$ is given by

$$\begin{aligned} c(\omega \oplus \phi) = h^* c(\gamma_1^m \oplus \gamma_2^n) &\in \mathbb{Z}[h^*(c_1(\gamma_1^m)), \ldots, h^*(c_m(\gamma_1^m)), h^*(c_1(\gamma_2^n)), \ldots, h^*(c_n(\gamma_2^n))] \\ &= \mathbb{Z}[c_1(h^*(\gamma_1^m)), \ldots, c_m(h^*(\gamma_1^m)), c_1(h^*(\gamma_2^n)), \ldots, c_n(h^*(\gamma_2^n))] \\ &= \mathbb{Z}[c_1(\omega), \ldots, c_m(\omega), c_1(\phi), \ldots, c_n(\phi)] \end{aligned}$$

This concludes the proof. ■

Before we begin the main proof, recall that a trivial bundle over B has the whole space B as a neighborhood for every local coordinate system.

Theorem 5.5. Let ω be a $\mathbb{C}^n$-bundle and ϕ a $\mathbb{C}^n$ bundle, both over B. Then $c(\omega \oplus \phi) = c(\omega)c(\phi)$.

Proof: As previously, we will prove this for canonical vector bundles γ^m, γ^n and extend to the general case. The proof will proceed by induction on $m+n$. The base case is immediate, so suppose that $c(\gamma^{m-1} \oplus \gamma^n) = c(\gamma^{m-1})c(\gamma^n)$, so

$$c(\gamma^{m-1} \oplus \gamma^n) = (1 + c_1(\gamma^{m-1}) + \cdots + c_{m-1}(\gamma^{m-1}))(1 + c_1(\gamma^n) + \cdots + c_n(\gamma^n)) \tag{1}$$

Let ε be the trivial line bundle over $Gr(m-1)$, and let $\gamma^{m-1} \oplus \varepsilon$ and γ^n be bundles over $Gr(m-1) \times Gr(n)$. By Lemma 5.4, we have that

$$c(\gamma^{m-1} \oplus \varepsilon \oplus \gamma^n) = p_{m,n}(c_1(\gamma^{m-1} \oplus \varepsilon), \ldots, c_m(\gamma^{m-1} \oplus \varepsilon), c_1(\gamma^n), \ldots, c_n(\gamma^n))$$

By Lemma 5.3, we have that $c_i(\gamma^{m-1} \oplus \varepsilon) = c_i(\gamma^{m-1})$ for all i, so

$$c(\gamma^{m-1} \oplus \gamma^n) = c(\gamma^{m-1} \oplus \varepsilon \oplus \gamma^n) = p_{m,n}(c_1(\gamma^{m-1}), \ldots, c_{m-1}(\gamma^{m-1}), 0, c_1(\gamma^n), \ldots, c_n(\gamma^n)) \tag{2}$$

For ease of notation, set $c_i = c_i(\gamma^{m-1})$ and $c'_j = c_j(\gamma^n)$ for all i, j. Compare equations (1) and (2) in this new notation for

$$p_{m,n}(c_1, \ldots, c_{m-1}, 0, c'_1, \ldots, c'_n) = (1 + c_1 + \cdots + c_{m-1})(1 + c'_1 + \cdots + c'_n)$$

Let c_m be a new indeterminate. Then in $\mathbb{Z}[c_1, \ldots, c_{m-1}, c_m, c'_1, \ldots, c'_n]$ we have that

$$p_{m,n}(c_1, \ldots, c_{m-1}, c_m, c'_1, \ldots, c'_n) \equiv (1 + c_1 + \cdots + c_{m-1} + c_m)(1 + c'_1 + \cdots + c'_n) \pmod{c_m}$$

Repeat the inductive step with $c(\gamma^m \oplus \gamma^{n-1})$ to get that, for some new indeterminate c'_n, in $\mathbb{Z}[c_1, \ldots, c_m, c'_1, \ldots, c'_{n-1}, c'_n]$,

$$p_{m,n}(c_1, \ldots, c_m, c'_1, \ldots, c'_{n-1}, c'_n) \equiv (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_{n-1} + c'_n) \pmod{c'_n}$$

The fact that c_m has been defined from the beginning here does not invalidate the first congruence, as it is presented modulo c_m. Note that $\mathbb{Z}[c_1, \ldots, c_{m-1}, c_m, c'_1, \ldots, c'_n]$ is a unique factorization domain, so

$$\begin{aligned} & p_{m,n}(c_1, \ldots, c_m, c'_1, \ldots, c'_n) \equiv (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n) \pmod{c_m c'_n} \\ \implies & p_{m,n}(c_1, \ldots, c_m, c'_1, \ldots, c'_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n) + q c_m c'_n \end{aligned}$$

for some $q \in \mathbb{Z}[c_1, \ldots, c_{m-1}, c_m, c'_1, \ldots, c'_n]$. However, $\dim(q) = 0$, as otherwise we would have $c_i(\gamma^{m-1} \oplus \varepsilon \oplus \gamma^n) \neq 0$ for some $i > 2(m+n)$, contradicting the definition of the Chern classes. So q is an integer. By the uniqueness in Lemma 5.4, we have that

$$c(\gamma^m \oplus \gamma^n) = p_{m,n}(c_1, \ldots, c_m, c'_1, \ldots, c'_n)$$

this proof is left unfinished

2 Additional material

2.1 Topology

Definition 1.1. Let ξ be a fiber bundle with projection map $\pi : E \to B$. Then a *section* of ξ is a continous map $s : B \to E$ such that for all $b \in B$, $\pi(s(b)) = b$.

2.2 Cellular and simplicial homology

The following definition is taken nearly verbatim from [3], page 118.

Definition 2.1. Let X be a topological space. Then X is termed a *CW-complex* if

$$X = \bigcup_{i=1}^{\infty} X^i \quad \text{where} \quad \begin{array}{rcl} X^0 & = & \text{a discrete space} \\ X^{i+1} & = & X^i \cup_{\varphi_i} \left(\bigsqcup_{\alpha \in \mathcal{A}_i} D_\alpha^{i+1} \right) \end{array} \quad \text{for} \quad \varphi_i : \bigsqcup_{\alpha \in \mathcal{A}_i} \partial D_\alpha^{i+1} \to X^i \ \text{ continuous}$$

The object D^i is the closed unit i-disk, with $D^i \subset X^i$ termed the *closed cell* of dimension i, and $\text{int}(D^i) \subset X^i$ termed the *open cell* of dimension i. The following conditions must also be satisfied:

1. each closed cell intersects finitely many open cells
2. $S \subset X$ is closed if and only if $S \cap D_\alpha^i$ is closed for all $\alpha \in \mathcal{A}_i$ and $i = 1, 2, \ldots$

Definition 2.2. A simplex is

Definition 2.3. relative homology

Suggested reading: [3]

2.3 Cohomology

Definition 3.1. The *cup product* is a product on cocycles, the elements of cohomology groups.

$$c^p \in C^p,\ c^q \in C^q \implies c^p \smile c^q \in C^{p+q}$$
$$\langle c^p \smile c^q, (v_0, \ldots, v_{p+q}) \rangle = \langle c^p, (v_0, \ldots, v_p) \rangle \cdot \langle c^q, (v_p, \ldots, v_{p+q}) \rangle$$

Definition 3.2. Let X, Y be topological spaces with natural projection maps:

$$X \times Y \xrightarrow{\pi_1} X \qquad\qquad X \times Y \xrightarrow{\pi_2} Y$$

These maps induce homomorphisms on the respective cochains groups over the base ring R:

$$C^*(X; R) \xrightarrow{\pi_1^*} C^*(X \times Y, R) \qquad\qquad C^*(Y; R) \xrightarrow{\pi_2^*} C^*(X \times Y, R)$$

Define the *cochain cross product*[1] $\times$ on $C^k(X; R) \otimes_R C^\ell(Y; R)$ by

$$\begin{tikzcd} C^k(X;R) \otimes_R C^\ell(Y;R) \arrow[r, "\pi_1^* \otimes \pi_2^*"] \arrow[rr, bend right, "\times"'] & C^k(X \times Y;R) \otimes_R C^\ell(X \times Y;R) \arrow[r, "\smile"] & C^{k+\ell}(X \times Y;R) \end{tikzcd}$$

We see that, given a k-cocycle $\varphi : C_k(X) \to R$ and an ℓ-cocycle $\psi : C_\ell(X) \to R$, the action is

$$\times : \varphi \otimes \psi \mapsto \pi_1^*(\varphi) \smile \pi_2^*(\psi)$$

The cross product may be extended to the cohomology groups $H^*(X; R)$ and $H^*(Y; R)$ in a canonical way.

Theorem 3.3. [Künneth]
Let X, Y be topological spaces.

Suggested reading: [4]

[1] see `http://folk.uio.no/rognes/kurs/mat4540h11/at2.pdf`, page 44 for the source of this definition

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[1] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer-Verlag, 1982.

[2] John Milnor and James Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[3] V.V. Prasolov. *Elements of Combinatorial and Differential Topology.* American Mathematical Society, 2006.

[4] V.V. Prasolov. *Elements of Homology Theory.* American Mathematical Society, 2007.

Index

Mathematicians