Graduate learning seminar on ∞-categories
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Contents

1 Simplicial sets

Jānis Lazovskis, 2017-09-11

Sources for this talk: [Gro15] Section 1.1, [GJ09] Chapter 1, [Rie11] Sections 2 and 3.

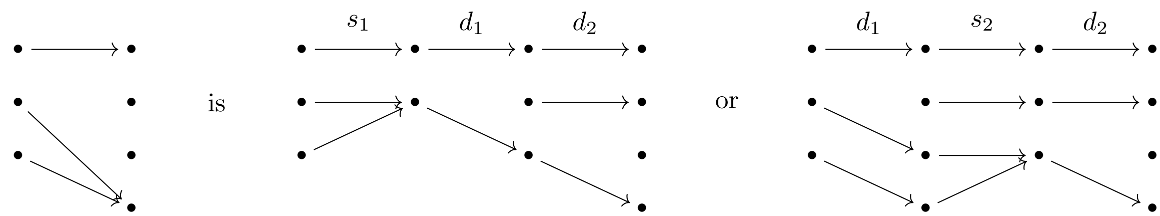
1.1 The category of simplicial sets

First we introduce some categories.

Δ: Objects are $[n] = (0, 1, \dots, n)$
Morphisms are non-decreasing (equivalently order-preserving) maps $[n] \to [m]$

Every morphism is a composition of: *coface maps* $s_i : [n] \to [n-1]$, hits i twice
codegeneracy maps $d_i : [n] \to [n+1]$, skips i

For example:



$[n]$: Objects are numbers $0, 1, \dots, n$
$|\mathrm{Hom}_{[n]}(a, b)| = 1$ iff $a \leqslant b$, else $\emptyset$

sSet: $\mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

An object (functor) may be described as $S = \{S_n = S([n])\}_{n \geqslant 0}$ with
face maps $S(s_i) : S_n \to S_{n+1}$
degeneracy maps $S(d_i) : S_n \to S_{n-1}$

Morphisms $f : S \to T$ are natural transformations

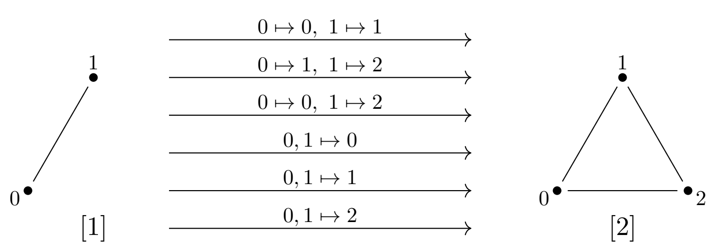
sObj: $\mathrm{Fun}(\Delta^{op}, C)$ for C any category

Remark 1. The final object in sSet, denoted by $*$, is the functor that takes everything to the empty set.

1.2 Examples of simplicial sets

Now we go through several examples of simplicial sets.

Example 1. $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$, called the *standard n-simplex*. For $n = 2$:



Check contravariance: $[0] \to [1]$ with $0 \mapsto 1$ should induce a map $\mathrm{Hom}_\Delta([1], [2]) \to \mathrm{Hom}_\Delta([0], [2])$. It does, by pre-composition of $(\alpha : [1] \to [2]) \in \mathrm{Hom}_\Delta([1], [2])$ by the same map $[0] \to [1]$.

Example 2. $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{top}, X)$, for X a topological space. Contravariance works as above.

Remark 2. There is an adjunction between the categories Top and sSet, given by geometric realization $| - |$ in one direction and $\mathrm{Sing}(-)$ in the other. This allows for a more visual representation of simplicial sets.

Example 3. $N(C)_n = \mathrm{Fun}([n], C)$, for C any category, called the *nerve* of C. Note that

$$\begin{array}{rcl} N(C)_0 & = & \text{objects of } C \\ N(C)_1 & = & \text{morphisms of } C \\ N(C)_2 & = & \text{pairs of composable arrows of } C \\ & \vdots & \\ N(C)_n & = & \text{strings of } n \text{ composable arrows of } C \end{array}$$

For example, the two maps $d_1, d_0 : [0] \to [1]$, given by $0 \mapsto 0$ and $0 \mapsto 1$, respectively, induce natural transformations from $F_1 : [1] \to C$ to $F_0 : [0] \to C$, which we may call the *domain* and *range*.

$$\begin{array}{lllcl} & & [0] \longrightarrow [1] & & (F_1 : [1] \to C) \longrightarrow (F_0 : [0] \to C) \\ \text{“domain”} & d_1 : & 0 \longmapsto 0 & \rightsquigarrow & (\alpha \in \mathrm{Hom}_C(A, B)) \longmapsto A \\ \text{“range”} & d_0 : & 0 \longmapsto 1 & & (\alpha \in \mathrm{Hom}_C(A, B)) \longmapsto B \end{array}$$

Note also that the face map $N(s_i) : \mathrm{Fun}([n], C) \to \mathrm{Fun}([n+1], C)$ inserts the identity arrow at the ith spot, and the degeneracy map $N(d_i) : \mathrm{Fun}([n], C) \to \mathrm{Fun}([n+1], C)$ composes the ith and $(i+1)$th arrows.

Example 4. $N([n]) = \Delta^n$, the standard n-simplex.

Example 5. Λ^n_i, the *ith n-horn*. Heuristically, it is generated by all elements of Δ^n except the ith face. Formally,

$$\Lambda^n_i = \bigcup_{\substack{\alpha \in \Delta^n \\ j \neq i}} d_j \circ s_j \circ \alpha.$$

There is a natural inclusion map (that is, inclusion natural transformation) $i : \Lambda^n_i \hookrightarrow \Delta^n$ for all $0 \leqslant i \leqslant n$.

1.3 Fibrations, Yoneda, and more examples

Before we talk about more examples, we need to introduce a new definition.

Definition 1. Let $f : S \to T$ be a morphism of simplicial complexes (that is, a natural transformation). Then f is a *fibration* if for every pair of morphisms $\Lambda^n_i \to S$ and $\Delta^n \to T$ such that the diagram on the left commutes, there exists a map $\Delta^n \to S$, so that the diagram on the right still commutes.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "i"'] & S \arrow[d, "f"] \\ \Delta^n \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "i"'] & S \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

Definition 2. An object $S \in \mathrm{sSet}$ is *fibrant*, or a *Kan complex*, if either of the equivalent conditions is satsifed:

- The canonical map $S \to *$ is a fibration.
- Every map $\Lambda^n_i \to S$ may be extended to a map $\Lambda^n_i \hookrightarrow \Delta^n \to S$.

Example 6. $\mathrm{Sing}(X)$ is a Kan complex, for X a topological space.

Example 7. $\Pi_1(X)$, the fundamental groupoid of a topological space X, is a Kan complex. Recall that objects are points in X and morphism from x to y is a homotopy classes of continuous maps $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Hence the relation to the fundamental group is

$$\pi_1(X, x) = \mathrm{Aut}_{\Pi_1(X)}(x).$$

It is a simplicial set because it is the nerve of the category of points of X and paths between points. Not sure why this is a Kan complex.

We finish with a useful statement.

Lemma 1. [Yoneda]
Let $S \in$ sSet. Morphisms $\Delta^n \to S$ in sSet (that is, natural transformations) correspond bijectively to elements of S_n. Moreover, the correspondence is natural in both directions.

In other words, $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, S) \cong S_n$. This statement makes it easier to describe morphisms between simplicial sets (that is, natural transformations).

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2 ∞-categories via Grothendieck

Micah Darrell, 2017-09-18

2.1 Motivation for higher categories

In "Pursuing Stacks," Grothendieck suggested there should a notion of "higher groupoids," with an "n-groupoid" modeling an "n-type."

1-groupoid: A category whose morphisms are all invertible. For example, a group is a groupoid with a single object (the set of elements of the group), and morphisms are multipliation by the group elements. We also have the *fundamental groupoid* $\Pi(X)$ for any topological space X, and the *classifying space* BG for any group G.

2-category: Contains objects, morphisms, and "morphisms between morphisms." For example, given $f : A \to B$, $g : B \to C$, and $h : C \to D$ all morphisms, we have two notions of a 2-category:
srict 2-category: if $h \circ (g \circ f) = (h \circ g) \circ f$
lax 2-category: if $h \circ (g \circ f) \cong (h \circ g) \circ f$

Theorem 1. [Mac Lane's coherence theorem]
Stict and lax 2-categories are the same, up to 2-categorical equivalence.

Issues arise in 3-categories. Lurie in [Lur09a] mentions that "no strict 3-groupoid can model S^2," where "to model" means to go functorially from an n-groupoid to a space. If functoriality were possible, it should be that the delooping of a groupoid should be the delooping of the space. Recall:

- The *delooping* of an n-groupoid G is an $(n+1)$-groupoid H with one object X, such that $\mathrm{Hom}_H(X, X) = G$.
- The *delooping* of a space X is a space F for which $X \cong \Omega F$.

Suppose we have an n-groupoid G, whose delooping is an $(n+1)$-groupoid H, a strict $(n+1)$-category. This gives a monoidal structure to G (meaning there is a functor $G \times G \to G$ satisfying some properties), call it $\nu : G \times G \to G$. If we can deloop H, then we get $\mu : H \times H \to H$, and this restricts to a new monoidal structure on G, call it $\mu_G : G \times G \to G$.The Eckmann–Hilton argument gives that G and H are commutative monoids, so μ_G and $|nu$ are the same, and G is a commutative monoid.

The above shows that an n-groupoid can be delooped infinitely many times, but a 2-sphere can only be delooped twice, confirming Lurie's statement.

Remark 3. To make things simpler (although losing some properties of higher morphisms), we use (∞, n)-categories to specify non-strict n-categories. We will be interested in $(\infty, 1)$-categories, which meand we have morphisms ar every level, and for $k > 1$ they are invertible.

2.2 Defining ∞-categories

Grothendieck's homotopy hypothesis says that $(\infty, 0)$-categories are spaces. Informally:

Definition 3. An $(\infty, 1)$-category is a category enriched in spaces.

Definition 4. A category $\mathcal{C}$ is *enriched* in a category $\mathcal{D}$ if

1. $\mathcal{D}$ has a monoidal structure,
2. $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{D}$, and
3. composition in $\mathcal{C}$ is given by the monoidal structure on $\mathcal{D}$.

This way of defining ∞-categories models some properties of spaces, but it is difficult to use. We can also relate an $(\infty, 1)$-category $\mathcal{C}$ to a 1-category $\mathrm{Ho}(\mathcal{C})$, called the *homotopy category* of $\mathcal{C}$, which has the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \Pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Recall from last time we had a space X and a Kan complex $\mathrm{Sing}(X)$, which meant that any map $\Lambda_i^n \to X$ from the ith n-horn can be lifted to a map $\Delta^n \to X$ from the n-simplex, for any n and all $0 < i < n$, called the *inner horn condition*. For any 1-category $\mathcal{C}$, we also had a simplicial set $N(\mathcal{C})$, the nerve, for which the lifting was unique. For example, if $n = 2$ and $i = 1$, compositions are unique, so the lifting is unique.

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

Definition 5. (after Quillen) An ∞-category is a simplicial set X with the inner horn condition.

Recall the *inner horn* is for $0 < i < n$, and the *outer horn* is for $i = 0, n$. Moreover if the inner horn condition gives a unique lifting, then the ∞-category is modeled by the nerve of some other category. We will denote the objects of an ∞-category $\mathcal{C}$ by $\mathcal{C}_0$, the morphisms by $\mathcal{C}_1$, and so on.

Note that horn liftings need not always be unique. For example, when $n = 2$ and $i = 1$, we can have

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] & & C \end{tikzcd} \quad \text{filling to} \quad \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "k"'] & & C \end{tikzcd} .$$

Definition 6. Two 1-simplices (arrows) $f, g : X \to Y$ are *homotopy equivalent* if there exists a 2-simplex

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "g"'] & & Y \end{tikzcd} .$$

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3 Examples of ∞-categories

Greg Taylor, 2017-09-26

Sources for this talk: [Fre13], [Lur09b], [AC16], [GR17], [Toe14], nLab article "Fundamental ∞-groupoid".

3.1 The fundamental ∞-groupoid

Let X be a topological space. Recall the functor $\mathrm{Sing}(X)$ that takes $[k] \in \Delta$ to $\mathrm{Hom}_{\mathrm{Top}}(|\Delta^k|, X)$. This is the simplicial set description of the *fundamental ∞-groupoid* $\Pi_\infty(X)$. Note the *fundamental groupoid* $\Pi_1(X)$ is the collection of 1-morphisms of $\Pi_\infty(X)$.

Remark 4. There is an adjunction $| \cdot | : \infty\text{-groupoid} \leftrightarrows \mathrm{Top} : \Pi_\infty(\cdot)$. This is the technical interpretation of the homotopy hypothesis.

3.2 Quasi-coherent sheaves

Now let X be a quasi-compact quasi-separated scheme. Without loss of generality, let X be a variety over an algebraically closed field k.

Definition 7. A *quasi-coherent* sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules for which X has an affine cover $\{\mathrm{Spec}(A_i)\}_i$ such that $\mathcal{F}|_{\mathrm{Spec}(A_i)} = \widetilde{M_i}$. Let $QC(X)$ be the category of quasi-coherent sheaves and sheaf morphisms on X.

Recall the notation of sheaves:

$$\widetilde{M_i}(D(f)) = (M_i)_f, \qquad (\widetilde{M_i})_p = (M_i)_p, \qquad D(f) = \{p \in \mathrm{Spec}(A_i) \ : \ f \notin p\} \subseteq \mathrm{Spec}(A_i).$$

Note that $QC(X)$ is symmetric monoidal, where the usual tensor product of sheaves gives it a symmetric monoidal structure.

Remark 5. Viewing $QC(X)$ as a subcategory of chain complexes, we get that $QC(X)$ is an ∞-category, call it $QC^\infty(X)$. Moreover, the two are related therough the homotopy category and derived categories. That is,

$$\mathrm{Ho}(QC^\infty(X)) = D(QC(X)).$$

Recall that the derived category $D(C)$ is constructed by first taking the category of complexes of C, then identifying chain homotopic morphisms, and finally localizing along quasi-isomorphisms (that is, quotienting by the quasi-isomorphism equivalence relation).

However, we cannot just "glue" derived categories to go from local to global properties. This does work in the ∞ setting. That is, if $\{\mathrm{Spec}(A_i)\}_i$ covers X, then

$$\mathrm{Ho}(QC^\infty(X)) = \mathrm{Ho}\left(\text{``}\lim_{\mathrm{Spec}(A_i)}\text{''} \ QC^\infty(\mathrm{Spec}(A_i))\right),$$

where the limit has not yet been defined in the ∞ setting.

3.3 TQFTs

Consider the category $\mathrm{Cob}(d \in \mathbf{Z}_{>0})$ of closed, oriented, compact $(d-1)$-manifolds and *cobordisms* between them.

Definition 8. Let M, N be objects of $\mathrm{Cob}(d)$. A *cobordism* (or *bordism*) between M and N is a d-manifold B for which $\partial B = M \sqcup N$.

Definition 9. A *topological quantum field theory* (or *TQFT*) is a functor $Z : \mathrm{Cob}(d) \to \mathrm{Vect}(\mathbf{C})$ with

- $Z(M \sqcup N) = Z(M) \otimes Z(N)$, and
- $Z(\emptyset) = \mathbf{C}$.

The above classical definition is due to Atiyah. Note that for any closed, compact d-manifold M without boundary, $Z(M : \emptyset \to \emptyset) = (\mathbf{C} \to \mathbf{C})$ is just a $\mathbf{C}$-endomorphism.

Theorem 2. [Cobordism Hyopthesis, Baez–Dolan]
Informally speaking, a TQFT is determined by its value on a point.

To prove this, Ayala, Francis, and Lurie have employed the extended category $\mathrm{Cob}(d)^{fr}_{ext}$ of framed cobordisms, which involves a choice of basis of the vector space in the target category. This is an ∞-category, with objects points (0-dimensional manifolds), 1-morphisms cobordisms between objects (1-dimensional manifolds), and d-morphsism cobordisms of $(d-1)$-manifolds (d-dimensional manifolds).

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4 Limits and colimits

Joel Stapleton, 2017-10-02

Sources for this talk: [Gro15], Sections 2.2 to 2.5.

4.1 Colimits in 1-categories

We wil only talk about final objects and colimits, everything can be dualized for initial objects and limits. Let $\mathcal{C}$ be a category.

Definition 10. An object X of $\mathcal{C}$ is *final* in $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(Y, X) = *$ for all objects Y in $\mathcal{C}$.

To extend colimits to higher categories, we define the category $\mathcal{C}_{p/}$, for every functor $p : \mathcal{D} \to \mathcal{C}$ of categories. This is called the *slice category*, or the *cocone of diagrams over* $\mathcal{C}$, and it may be viewed as a subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$. For example, suppose we have a diagram

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$$

in $\mathcal{D}$. Apply the functor p to this diagram to get a similar diagram in $\mathcal{C}$, whose cocone

$$\begin{tikzcd} & \bullet \arrow[dr, dashed] & \\ \bullet \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & \bullet \\ & \bullet \arrow[ur, dashed] & \end{tikzcd}$$

is another object of $\mathcal{C}$ and morphisms into that object, so that the diagram commutes.

Definition 11. A *colimit* of p is an initial object in $\mathcal{C}_{p/}$.

Actually, it is not quite an initial object, but the composition of the initial object with the constant functor that gives the target object of a morphism.

4.2 The join construction in 1-categories

Let A, B be categories.

Definition 12. The *join* $A \star B$ of A and B is another category, for which

$$\begin{aligned} \mathrm{obj}(A \star B) &= \mathrm{obj}(A) \coprod \mathrm{obj}(B), \\ \mathrm{hom}(X, Y) &= \begin{cases} \mathrm{hom}_A(X, Y) & \text{if } X, Y \in \mathrm{obj}(A), \\ \mathrm{hom}_B(X, Y) & \text{if } X, Y \in \mathrm{obj}(B), \\ * & \text{if } X \in \mathrm{obj}(A), Y \in \mathrm{obj}(B), \\ \emptyset & \text{else.} \end{cases} \end{aligned}$$

The idea is to think of this as connecting two categories together with morphisms.

Example 8. For $\mathbf{1}$ the terminal object in the category of small categories and $\mathcal{C}$ any other category, the join $\mathcal{C} \star \mathbf{1}$ is simply the cocone of $\mathcal{C}$, and $\mathbf{1} \star \mathcal{C}$ is the cone of $\mathcal{C}$.

The join is not symmetric in general, but it is symmetric in the geometric interpretation.

Example 9. Consider two simplices $[i]$ and $[j]$ as categories (as defined in Section 1.1). Their join is

$$[i] \star [j] = [i + j + 1].$$

Let us check this for $i = j = 0$. We begin with two 0-simplices

$$\begin{tikzcd} \bullet \arrow[loop above] & \bullet \arrow[loop above] \end{tikzcd}$$

and add on a single element for a morphism between them, giving

$$\begin{tikzcd} \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \end{tikzcd}$$

which is indeed the 1-simplex [1].

We are trying to get a universal characterization of the slice category. The following statement gives us that.

Proposition 1. $\mathrm{Fun}(A, \mathcal{C}_{/p}) \cong \mathrm{Fun}_p(A \star \mathcal{D}, \mathcal{C})$.

The category on the right may be viewed as all the morphisms that makes diagram of the type

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "i"'] \arrow[dr, "p"] & \\ A \star \mathcal{D} \arrow[rr, dashed] & & \mathcal{C} \end{tikzcd}$$

commute.

4.3 The join construction in simplicial sets

Now we translate joins into the laguage of simplicial sets, which should account for all the higher dimensional objects. Let K, L be simplicial sets, or elements of obj(sSet).

Definition 13. The *join* $K \star L$ of K and L is another simplicial set, for which

$$(K \star L)_n = K_n \cup L_n \bigcup_{i+j+1=n} K_i \times L_j.$$

Example 10. Let us check this for small n. When $n = 0, 1$, we have

$$(K \star L)_0 = K0 \cup L_0, \qquad (K \star L)_1 = K_1 \cup L_1 \cup (K_0 \times L_0).$$

To take this to the subcategory of ∞-categories, we need to make sure it is closed and preserves equivalences there. We cite a proposition from [Gro15].

Proposition 2.
1. *If $\mathcal{C}, \mathcal{D}$ are ∞-categories, then $\mathcal{C} \star \mathcal{D}$ is an ∞-category.*
2. *If $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{C}' \to \mathcal{D}'$ are equivalences of ∞-categories, then $f \star g$ is also an equivalence of ∞-categories.*

The map $f \star g : \mathcal{C} \star \mathcal{C}' \to \mathcal{D} \star \mathcal{D}'$ is defined in the natural way. An *equivalence* of ∞-categories may be thought of a particular morphism of simplicial sets (which is a natural transformation between two functors out of Δ^{op}), although defining it precisely requires the notion of an ∞-*morphism*, which we do not attempt to describe here.

Proposition 3. $N(A \star B) \cong N(A) \star N(B)$.

Definition 14. Let $p : L \to \mathcal{C}$ be a map of simplicial sets, with $\mathcal{C}$ an ∞-category. There is an ∞-category $\mathcal{C}_{/p}$ characterized by the following universal property: For any $K \in \mathrm{obj(sSet)}$,

$$\hom_{\mathrm{sSet}}(K, \mathcal{C}) \cong \hom_{\mathrm{sSet}\ L/}(L \to K \star L, L \xrightarrow{p} \mathcal{C}).$$

For example, $(\mathcal{C}_{/p})_n = \hom_{\mathrm{sSet}}([n], \mathcal{C}_{/p}) = \hom_{\mathrm{sSet}\ L/}(L \to [n] \star L, L \xrightarrow{p} \mathcal{C})$.

Lemma 2. *If $p : A \to B$ is a morphsim of categories, then $N(B_{/p}) \cong N(B)_{/p}$.*

Proof. (Sketch) Observe that

$$\begin{aligned} N(B_{/p})_n &\cong \hom_{\mathrm{sSet}}([n], N(B_{/p})) \\ &\cong \hom_{\mathrm{Cat}}([n], B_{/p}) && \text{(by full faithfulness of hom-sets)} \\ &\cong \hom_{\mathrm{Cat}\ A/}(A \to [n] \star A, A \to B) && \text{(by characterization of join)} \\ &\cong \hom_{\mathrm{sSet}\ N(A)/}(N(A) \to [n] \star N(A), N(A) \xrightarrow{N(p)} N(B)) && \text{(take the nerve)} \\ &\cong \hom_{\mathrm{sSet}}([n], N(B)_{/N(p)}). \end{aligned}$$

□

We now need to describe what final objects are in ∞-categories.

4.4 Final objects in ∞-categories

Definition 15. An object X in an ∞-category $\mathcal{C}$ is *final* if $\mathcal{C}_{/X} \to \mathcal{C}$ is an acyclic fibration of simplicial sets.

An immediate corollary of this definition is the following:

Proposition 4. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory of final objects. Then $\mathcal{D}$ is either empty or a contractible Kan complex.*

Definition 16. Let K be a simplicial set and $\mathcal{C}$ an ∞-category. A *colimit* of a morphism $p : K \to \mathcal{C}$ is an initial object of $\mathcal{C}_{p/}$.

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5 Model categories

Harry Smith, 2017-10-09

Sources for this talk: [DS95]

Newt week we will see an equivalence between presentable ∞-categories and combinatorial model categories, this week will be setting up all the necessary definitions.

5.1 The model category axioms

Before we can give the axioms for a model category, we need some preliminary definitions.

Definition 17. A morphism $f \in \mathrm{Hom}(A, B)$ is a *retract* of $g \in \mathrm{Hom}(C, D)$ if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r] & B \end{tikzcd}$$

commutes. Further, f has the *left lifting property* (LLP) with respect to g, and g has the *right lifting property* (RLP) with respect to f if for every commuting diagram without the dashed arrow, the dashed arrow exists, keeping commutativity.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

Now we have enough to define model categories.

Definition 18. A *model category* is a category $\mathcal{C}$ with three classes F (denoting *fibrations* $\twoheadrightarrow$), C (denoting *cofibrations* $\hookrightarrow$), and W (denoting *weak equivalences* $\xrightarrow{\sim}$) of distinguished morphisms, satisfying the conditions below. Elements of $F \cap W$ are called *acyclic fibrations* $\overset{\sim}{\twoheadrightarrow}$ and elements of $C \cap W$ are called *acyclic cofibrations* $\overset{\sim}{\hookrightarrow}$.

1. $\mathcal{C}$ is complete and cocomplete (limits and colimits exist).

2. Weak equivalences satisfy the two-out-of-three rule. That is, if any of two of the thee arrows in the commuting diagram

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

of objects and morphisms in $\mathcal{C}$ are weak equivalences, then the third one is as well.

3. Whenever f is a retract of g, if g is in F, C, W, then f is in F, C, W, respectively.
4. A cofibration has the LLP with respect to any acyclic fibration. A fibration has the RLP with respect to any acyclic cofibration.
5. Every morphism $\bullet \longrightarrow \bullet$ can be factored as an acyclic cofibration followed by a fibration, or a cofibration followed by an acyclic fibration. That is, we have the following commutative diagram.

$$\begin{tikzcd}
 & \bullet \arrow[dr, two heads] & \\
\bullet \arrow[ur, hook, "\sim"] \arrow[rr] \arrow[dr, hook] & & \bullet \\
 & \bullet \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

5.2 Fibrations and cofibrations

The initial object of a model category $\mathcal{C}$ is denoted $\emptyset$ and the final object is $*$.

Definition 19. An object A of a model category $\mathcal{C}$ is *fibrant* if $A \twoheadrightarrow *$ and *cofibrant* if $\emptyset \hookrightarrow A$. That is, if the unique maps into $*$ and out of $\emptyset$ are fibrations and cofibrations, respectively. An object that is both fibrant and cofibrant is called *bifibrant*. The unique objects A_F and A_C that make the middle three levels of the diagram

$$\begin{tikzcd}
 & & \emptyset \arrow[dl, hook] \arrow[dr] & & \\
 & A_{CF} \arrow[rr, two heads, "\sim"] & & A_F \arrow[dr, two heads] & \\
\emptyset \arrow[rr] \arrow[dr, hook] & & A \arrow[ur, hook, "\sim"] \arrow[rr] & & * \\
 & A_C \arrow[ur, two heads, "\sim"] \arrow[rr, hook, "\sim"] \arrow[dr] & & A_{CF} \arrow[dl, two heads] & \\
 & & * & &
\end{tikzcd}$$

commute are called the *fibrant replacement* of A and the *cofibrant replacement* of A, respectively. A *bifibrant replacement* A_{CF} of A is the fibrant replacement of the cofibrant replacement, or, weakly equivalently, the cofibrant replacment of the fibrant replacement.

The existence of the fibrant and cofibrant replacements is guaranteed by axiom 5.

Proposition 5. *In a model category $\mathcal{C}$:*
1. *Cofibrations are exactly those with the LLP with respect to all acyclic fibrations.*
2. *Acyclic cofibrations are exactly those with the LLP with respect to all fibrations.*
3. *Fibrations are exactly those with the RLP with respect to all acyclic cofibrations.*
4. *Acyclic fibrations are exactly those with the RLP with respect to all cofibrations.*

Let A be an object of a model category $\mathcal{C}$. Since $\mathcal{C}$ is complete and cocomplete, we have the product $A \Pi A$ and coproduct $A \amalg A$, with natural maps out of A and into A, respectively.

Definition 20. A *cylinder object* $A \wedge I$ and *path object* A^I are the objects that make the diagram

$$\begin{tikzcd}
 & & & A^I \arrow[dr, two heads] & \\
A \amalg A \arrow[rr, "(\mathrm{id}, \mathrm{id})"] \arrow[dr, hook] & & A \arrow[ur, hook, "\sim"] \arrow[rr, "(\mathrm{id}, \mathrm{id})"'] & & A \Pi A \\
 & A \wedge I \arrow[ur, two heads, "\sim"] & & &
\end{tikzcd}$$

commute. For $f, g : A \to B$ and $h, k : B \to A$ if the cofibrant replacement of $(f, g) : A \amalg A \to B$ is $A \wedge I$, then f and g are *left homotopic*, denoted $f \overset{\ell}{\sim} g$. If the fibrant replacement of $(h, k) : B \to A \Pi A$ is B^I, then h and k are *right homotopic*, denoted $g \overset{r}{\sim} k$. That is, $f \overset{\ell}{\sim} g$ and $g \overset{r}{\sim} k$ if the following diagram commutes.

$$\begin{tikzcd}
& & & B^I \arrow[dr] & \\
A \amalg A \arrow[rr, "(f,g)"] \arrow[dr] & & B \arrow[ur, "\sim"] \arrow[rr, "(h,k)"'] & & A \Pi A \\
& A \wedge I \arrow[ur, "\sim"] & & &
\end{tikzcd}$$

If $f \overset{\ell}{\sim} g$ and $f \overset{r}{\sim} g$, then f and g are simply *homotopic*.

As before, the existence of cylinder and path objects is guaranteed by axiom 5. Also note that the smash product does not necessarily exist in a model category, we simply use the notation for intuition from topological spaces.

Proposition 6. *Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.*
1. *If A is cofibrant, then $f \overset{\ell}{\sim} g$ implies $f \overset{r}{\sim} g$.*
2. *If B is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{\ell}{\sim} g$.*
3. *If A, B are both fibrant, then f is a homotopy equivalence iff f is a weak equivalence.*

We use the term *homotopy equivalence* to mean f is homotopic to the identity map. We now revisit an earlier definition.

Definition 21. The *homotopy category* of a model category $\mathcal{C}$ has the same objects as $\mathcal{C}$, with

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A_{CF}, B_{CF})/(\text{weak equivalences}).$$

We can equivalently mod out by homotopy equivalences. Note that we may think of the natural functor $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ as a localization inverting weak equivalences.

5.3 Examples

Now we consider model structures on some familiar categories.

Example 11. Top is a model category:
- The weak equivalences are homotopy equivalences.
- The fibrations are Hurewicz fibrations (maps that have the RLP with respect to $A \hookrightarrow A \times I$).
- Ho(Top) is the homotopy category of topological spaces

Example 12. Top is a model category in a different way:
- The weak equivalences are weak equivalences (maps that induce equivalences $\pi_i(A) \xrightarrow{\cong} \pi_i(B)$).
- The cofibrations are retracts of attaching cells.
- Ho(Top) is the homotopy category of CW complexes.

Example 13. Ch_R, the category of chain complexes over a positively graded ring R, is a model category:
- The weak equivalences are quasi-isomorphisms.
- The cofibrations are degree-wise monomorphisms.
- $\mathrm{Ho}(\mathrm{Ch}_R)$ is the derived category of $\mathbf{R}$.

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6 More model categories and presentable categories

Harry Smith, 2017-10-16

The motivation for this talk is the *adjoint functor theorem*, which states that a functor F of cocomplete categories is a left adjoint iff F preserves colimits and satisfies the solution set condition. Having presentable categories allows us to get rid of the solution set condition.

6.1 Vocabulary buildup

We need to define several things before we get to presentable categories. Let $\mathcal{C}$ be a category.

Definition 22. A *filtered system* is a functor $I \to \mathcal{C}$ from the index category such that:
- every pair of objects has another object both map into, and
- every pair of morphisms between the same objects has a third morphism that makes them equal.

That is, for every diagram without the dashed arrows, the dashed arrows exist.

$$\begin{tikzcd} \bullet \arrow[dr, dashed] & \\ & \bullet \\ \bullet \arrow[ur, dashed] & \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, dashed] & \bullet \end{tikzcd}$$

For κ a cardinal and $A \subset I$ with $|A| < \kappa$ in objects and morphisms, a κ*-filtered system* is one where adjoining the final object to form A_0 makes the diagram below commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & I \\ A_0 \arrow[ur, dashed] & \end{tikzcd}$$

A filtered system is a κ-filtered system for $\kappa = \aleph_0$. Also note that if $\kappa < \lambda$, then a λ-filtered system is immediately κ-filtered.

Definition 23. A category $\mathcal{C}$ is κ*-accessible* if it admits κ-filtered colimits and there is a small $\mathcal{D} \subset \mathcal{C}$ such that
- every $a \in \mathcal{C}$ is a κ-filtered colimit of a system in $\mathcal{D}$, and
- $\hom_{\mathcal{C}}(d, -)$, for $d \in \mathcal{D}$, preserves κ-filtetred colimits.

Definition 24. A category $\mathcal{C}$ is *(locally) presentable* if it is κ-accessible for some κ and cocomplete. It is κ*-accessible* if $\mathcal{C} \to \mathcal{D}$, for $\mathcal{D}$ as above, preserves κ-filtered colimits.

Example 14. Here are some example of presentable categories:
- Set is presentable, with $\mathcal{D} = \{*\}$.
- $\text{Fun}(A^{op}, \text{Set})$ for A small is presentable. Recall for $A = \Delta$ this is sSet.
- $\text{Ring}(R)$, $\text{Mod}(R)$, $\text{Ch}(R)$ are all presentable.
- quasi-coherent $\mathcal{O}_X$-modules over a scheme X is presentable.

We now return to the original motivation.

Theorem 3. [Adjoint functor theorem]
Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of presentable 1-categories. Then:
- *F is left adjoint iff F preserves colimits, and*
- *F is right adjoint iff F preserves limits and is κ-accessible for some $\kappa > \aleph_0$.*

6.2 The ∞-Yoneda embedding

Now let $\mathcal{C}, \mathcal{D}$ be ∞-categories.

Definition 25. A *localization* of ∞-categories is a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint. An *accessible localization* is a localization whose right adjoint is accessible.

Theorem 4. *If a 1-category $\mathcal{C}$ is presentable, then $\mathcal{C}$ is equivlent to an accessible localization of* $\text{Fun}(A^{op}, \text{Set}$*, for some A small.*

The above theorem should be taken in the context of the Yoneda embedding. That is, let the first arrow below be the cocompletion and the second arrow be the accessible localization.

$$A \to \text{Fun}(A^{op}, \text{Set}) \to \mathcal{C}.$$

The Yoneda embedding in the ∞ setting uses the coherent nerve functor and the adjunction $\mathcal{C}[-], N_\Delta : \mathrm{sSet} \leftrightarrows \mathrm{sCat}$, for sCat the simplicially enriched category, where the hom-sets are simplicial sets. For some $K \in \mathrm{sSet}$, we have a map

$$\begin{aligned} \mathrm{Map} : \mathcal{C}[K]^{op} \times \mathcal{C}[K] &\to \mathrm{sCat}, \\ (X, Y) &\mapsto \hom(\Delta \times X, Y). \end{aligned}$$

This gives maps

$$\mathcal{C}[K^{op} \times K] \xrightarrow{\text{canonical}} \mathcal{C}[K^{op}] \times \mathcal{C}[K] \to \mathrm{sCat} \xrightarrow{\text{fibrant replacement}} \mathrm{Kan},$$

where Kan is the category of Kan complexes. Adjunction given us a map $K^{op} \times K \to N_\Delta(\mathrm{Kan}) = \mathcal{S}$, the exponential adjunction. The ∞-Yoneda embedding then is $K \to \mathrm{Fun}(K^{op}, \mathcal{S}) = P(K)$. This allows us to restate the previous theorem as follows.

Theorem 5. *An ∞-category $\mathcal{C}$ is presentable iff is it an accessible localization of $P(K)$, for some small K.*

6.3 Relation to ∞-categories

Definition 26. A *combinatorial model category* is one that
- is presentable as a 1-category, and
- is cofibrantly generated.

A category is *cofibrantly generated* if there is a set (as opposed to a class) of cofibrations generating the class of cofibrations.

Example 15. The categories sSet and $\mathrm{Fun}(A^{op}, \mathrm{sSet})_{\mathrm{proj}}$ are combinatorial model categories.

Theorem 6. *An ∞-category $\mathcal{C}$ is presentable iff there exists a simplicial combinatorial model category M with $\mathcal{C} = N_\Delta(M_{CF})$.*

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7 Straightening and unstraightening

Micah Darrell, 2017-10-23

Sources for this talk: [Lur09a], [GR17] Chapter I.1.4.

There is an equivalence of ∞-categories $\mathrm{Cart}_{/\mathcal{C}}$ and $\mathrm{Fun}_\infty(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$, where Cart will be described later. The forward direction is *straightening* and the backward direction is *unstraightening*. The point of this is that $\mathrm{Cart}_{/\mathcal{C}}$ is a purely $(\infty, 1)$-object, whereas Cat_∞ is an $(\infty, 2)$-category. This identifiation will do at least two things for us:

1. If we want to construct a functor $F : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$, we can equivalently construct a functor $p : G \to \mathcal{C}$ and check that p is a Cartesian fibration.
2. If we have a diagram $F : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$, we can "unstraighten" this to a functor $p : G \to \mathcal{C}$, and then complete $\varprojlim F$ through p.

7.1 The Grothendieck construction

Definition 27. A *category over* $\mathcal{C}$ is the data of a category F and a functor $p : F \to \mathcal{C}$. If $p : F \to \mathcal{C}$ is a category over $\mathcal{C}$, an *arrow* $\varphi : d_0 \to d_1$ in F is called *p-cartesian* if for any object d' and any map $\lambda : d' \to d_1$ such that $F(\lambda) = g \circ f$, there exists a unique $\psi : d' \to d_0$ such that $F(\psi) = g$. That is, given the solid arrows in the diagrams below, the dotted arrow exists, keeping commutativity.

$$\begin{tikzcd}
d' \arrow[r, dashed, "\psi"] \arrow[d] \arrow[rr, bend left, "\lambda"] & d_0 \arrow[r, "\varphi"] \arrow[d] & d_1 \arrow[d] \\
p(d') \arrow[r, "g"] \arrow[rr, bend right] & p(d_0) \arrow[r, "f"] & p(d_1)
\end{tikzcd}$$

Definition 28. A *fibered category* over $\mathcal{C}$ is a category over $\mathbf{C}$, $p : F \to \mathcal{C}$, such that for every morphism $g : U \to V$ and $|mathcalC$ and $v \in p^{-1}(V)$, there exists a Cartesian arrow $\varphi : u \to v$ such that $p(\varphi) = g$.

So in particular, there exists $u \in p^{-1}(U)$.

7.2 The category $F(U)$

Now we define a new category. For every object $U \in \mathcal{C}$, let $F(U)$ be the category whose objects are $u \in F$ such that $p(u) = U$, and morphisms are $f : u \to v$ such that $p(f) = \mathrm{id}_U$, so $\mathrm{obj}(F(U)) = p^{-1}(U) \subseteq \mathrm{obj}(F)$. The arrows in $F(U)$ are arrows in F that p sends to id_U.

Remark 6. Applying the axiom of choice, the functor $p : F \to \mathcal{C}$ is equivlent to a functor $\mathcal{C}^{op} \to \mathrm{Cat}$:

$$\begin{array}{rcl} \mathcal{C}^{op} \ni U & \mapsto & F(U) \in \mathrm{Cat}, \\ (g : U \to V) & \mapsto & \text{(chocie of Cartesian map in } F). \end{array}$$

Recall that a Cartesian map in F is a functor $F(V) \to F(U)$. The idea is to consider the following diagram:

$$\begin{tikzcd} F(U) \arrow[d] & F(V) \arrow[l] \arrow[d] \\ U \arrow[r, "g"] & V \in \mathcal{C} \end{tikzcd}$$

So, $p : F \to \mathcal{C}$ is a Cartesian fibration if for every $(g : U \to V) \in \mathcal{C}$ and $v \in p^{-1}(V)$, there exists a Cartesian arrow $\varphi : u \to v$ in F such that $p(\varphi) = g$. So, in particular, $u \in p^{-1}(U)$.

Example 16. The functor from spaces to the categroy of categories is a cartesian fibration:

$$\begin{array}{rcl} \mathcal{S} & \to & \mathrm{Cat}, \\ X & \mapsto & \mathrm{Vect}(X), \\ (f : X \to Y) & \mapsto & (f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)), \end{array}$$

where f^* is the pullback of vector bundles.

If we choose for every $f \in \mathcal{C}$ a Cartesian arrow in F, then this data specifies a functor $U \to F(U)$, with $(f : U \to V) \mapsto (F^* F(V) \to F(U))$. The punchline of this whole talk is that the $(\infty, 1)$-Grothendieck construction does the same thing with Cartesian fibrations of $(\infty, 1)$-categories.

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8 Constructible sheaves and exit paths

Jānis Lazovskis, 2017-10-30

Sources for this talk: [Lur09a] Chapter 1, [Lur16] Appendix A, [Gro15] Section 2.

8.1 Locally constant sheaves

Let X be a topological space and $Op(X)$ the category of open sets (and inclusions) of X. Recall:
- $\mathcal{K}$an is the full subcategory of sSet spanned by Kan complexes (lifting property),
- $\mathcal{S} = N(\mathcal{K}\mathrm{an})$ is the ∞-category of spaces, given as the nerve.

The simplicial nerve $N(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is an ∞-category whenever the mapping spaces between objects of $N(\mathcal{C})$ are Kan complexes.

Definition 29. A *presheaf* $\mathcal{F}$ on X is a functor $\mathcal{F} : Op(X)^{op} \to \mathrm{Set}$. A *sheaf* $\mathcal{F}$ on X is a presheaf that satisfies the *gluing condition*: given a cover $\{U_i\}_{i \in I}$ of X and a collection $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$,

$$\Big(\mathcal{F}(U_i \cap U_j \hookrightarrow U_i)(s_i) = \mathcal{F}(U_i \cap U_j \hookrightarrow U_j)(s_j) \ \forall\, i, j \in I\Big) \quad \Longrightarrow \quad \Big(\exists!\ s \in \mathcal{F}(X) \ : \ \mathcal{F}(U_i \hookrightarrow X)(s) = s_i \ \forall\, i \in I\Big).$$

The *stalk* of $\mathcal{F}$ at $x \in X$ is $\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$.

This definition may be generalized to a sheaf on a category $\mathcal{C}$ valued in the ∞-category of spaces $\mathcal{S}$. However, this requires the deinfinition of a *site* and *coverage* of a category, which is not relevant here.

Example 17. Some common examples of sheaves are:

- the *constant sheaf* $\mathcal{F}(U) = A$ for some $A \in \text{Set}$ and all $U \subseteq X$,
- the *locally constant sheaf* $\mathcal{F}$: every $x \in X$ has a neighborhood $U \ni x$ such that $\mathcal{F}|_U$ is constant.

Let $\text{Shv}(X)$ be the category of sheaves on X, and $\text{Shv}_{LC}(X) \subseteq \text{Shv}(X)$ the category of locally constant sheaves on X (a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is a natural transformation).

Proposition 7. $\text{Shv}_{LC}(X) \cong \mathcal{S}_{/\text{Sing}(X)}$*, and so is an* ∞*-category.*

Proof. (sketch) Remark A.1.4 of [Lur16] gives that $\text{Shv}(X)$ has *locally constant shape* if X is a paracompact topological space with the homotopy type of a CW complex. Because of this Theorem A.1.15 of [Lur16] implies that $\text{Shv}(X) \cong \mathcal{S}_{/K}$ for some fixed $K \in \mathcal{S}$. Since the *shape* of $\text{Shv}(X)$ can be identified with $\text{Sing}(X)$, it follows that $K = \text{Sing}(X)$.

Finally, Proposition 2.17 of [Gro15] gives that the slice category $\mathcal{S}_{/\text{Sing}(X)} := \mathcal{S}_{/p}$, for $p : \Delta^0 \to \mathcal{S}$ given by $* \mapsto \text{Sing}(X)$, is an ∞-category. □

This proof is not very satisfying. Some new terms, for $\mathcal{X}$ a topos:
- the *shape* of $\mathcal{X}$ is the functor $\pi_* : \mathcal{X} \to \text{Set}$ corepresented by the final object $\{*\}$ in $\mathcal{X}$
- $\mathcal{X}$ has *constant shape* if $\pi_*\pi^* : \text{Set} \to \text{Set}$ is corepresentable
- $X \in \mathcal{X}$ has *constant shape* if $\mathcal{X}_{/X}$ has constant shape
- $\mathcal{X}$ has *locally constant shape* if every $X \in \mathcal{X}$ has constant shape

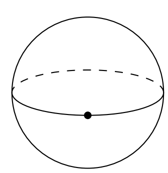
Now we generalize locally constant sheaves to *constructible* sheaves.

8.2 Constructible sheaves

Definition 30. Let $(A, \leqslant)$ be a poset. An A-*stratification* (or simply *stratification* when A is clear) is a continuous map $f : X \to A$, where A is given the upset topology ($x \in U$ and $x \leqslant y$ implies $y \in U$).

If X has a stratification, then X is called a *stratified space*. Given a stratification $f : X \to A$, the space $X_a = \{x \in X \ : \ f(x) = a\}$ is called a *level set*. Objects $X_{>a}$, $X_{\geqslant a}$ are defined analogously.

Example 18. Consider the following examples of A-stratifications.
- Let $X = S^2$ and $A = \{0 \leqslant 1 \leqslant 2\}$. Pick a point X_0 and a great circle $X_0 \cup X_1$ through that point.



- Let X be a simplicial complex and $A = \mathbf{Z}_{\geqslant 0}$. Let X_a be the a-skeleton of X.

Definition 31. Let $f : X \to A$ be a stratified space. An element $\mathcal{F} \in \text{Shv}(X)$ is *constructible* if $\mathcal{F}|_{X_a} \in \text{Shv}_{LC}(X_a)$ for all $a \in A$. The category of A-constructible sheaves is denoted $\text{Shv}^A(X)$.

Note that $\text{Shv}_{LC}(X) \subseteq \text{Shv}^A(X)$ for all A, and $\text{Shv}_{LC}(X) = \text{Shv}^{\{*\}}(X)$. By Proposition A.5.9 of [Lur16], every element of $\text{Shv}^A(X)$ is *hypercomplete* as an object of $\text{Shv}(X)$, which implies that $\text{Shv}^A(X)$ is an ∞-category.

8.3 Exit path equivalence

Now we describe A-constructible sheaves in a different way.

Definition 32. An *exit path* in a stratified space $f : X \to A$ is a continuous map $\gamma : [0,1] \to X$ for which there exists a pair of chains $a_1 \leqslant \cdots \leqslant a_n$ in A and $0 = t_0 \leqslant \cdots \leqslant t_n = 1$ in $[0,1]$ such that $f(\gamma(t)) = a_i$ whenever $t \in (t_{i-1}, t_i]$.

This really is a path, and so gives good intuition for what is happening. Recall that the *geometric realization* of the n-simplex Δ^n is $|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} \ : \ t_0 + \cdots + t_n = 1\}$. Oserving that $[0,1] \cong |\Delta^1|$, this definition may be generalized by instead considering maps from $|\Delta^n|$.

Definition 33. An *exit path* in a stratified space $f : X \to A$ is a continuous map $\gamma : |\Delta^n| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A such that $f(\gamma(t_0, \ldots, t_i, 0, \ldots, 0)) = a_i$ for $t_i \neq 0$.

Let $\mathrm{Sing}^A(X) \subseteq \mathrm{Sing}(X)$ be the category of exit paths on $f : X \to A$. If $f : X \to A$ is *conically stratified*, then $\mathrm{Sing}^A(X)$ is an ∞-category (Theorem A.6.5 in [Lur16]). This means that every $x \in X$ has a neighborhood that looks like $Z \times C(Y)$, for appropriate topological spaces Z, Y (and $C(Y)$ the cone of Y).

Lemma 3. *If X has locally singular shape, then* $\mathrm{Shv}_{LC}(X) \cong \mathrm{Fun}(\mathrm{Sing}(X), \mathcal{S})$.

It is sufficient for X to be locally contractible for X to have locally singular shape. This may be viewed as straightening and unstraightening (see Section 7).

This lemma combines Proposition 7 above and Theorem A.1.15 in [Lur16], as $\mathrm{Sing}(X)$ is the shape of $\mathrm{Shv}(X)$. Citing directly from Section A.6 of [Lur16], the idea now is that a sheaf $\mathcal{F} \in \mathrm{Shv}_{LC}(X)$ can be interpreted as a functor which assigns to each $x \in X$ the stalk $\mathcal{F}_x \in \mathcal{S}$ and to each path joining x to y the homotopy equivalence $\mathcal{F}_x \cong \mathcal{F}_y$. This generalizes to:

Theorem 7. *If X has locally singular shape and $f : X \to A$ is conically stratified, then* $\mathrm{Shv}^A(X) \cong \mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S})$.

We now follow Construction A.4.17 and its variants in Section A.9 of [Lur16]. It should be noted the first source uses $\mathcal{P}(\mathcal{U}(X))$ to mean presheaves on open set of X and the second uses $\mathcal{P}(\mathcal{B}(X)$ to mean presheaves on open F_σ subsets of X, but we conflate the notation here.

$$
\begin{array}{rcll}
(U, Y) & \mapsto & \mathrm{Fun}_{\mathrm{Sing}(X)}(\mathrm{Sing}(U), Y) & \\
\theta \ : \ Op(X)^{op} \times \mathrm{sSet}_{/\mathrm{Sing}(X)} & \to & \mathrm{sSet} & \\
\theta|_{bifib} \ : \ Op(X)^{op} \times \mathbf{A}_X^o & \to & \mathcal{K}\mathrm{an} & \text{take bifibrant objects} \\
N(\theta|_{bifib}) \ : \ N(Op(X)^{op}) \times N(\mathbf{A}_X^o) & \to & \mathcal{S} & \text{take nerve} \\
\psi \ : \ N(\mathbf{A}_X^o) & \to & p\mathrm{Shv}(X) & \text{equivalently} \\
\psi \ : \ \mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) & \to & p\mathrm{Shv}(X) & \text{Notation A.9.1}
\end{array}
$$

Above, we let $\mathbf{A}_X$ denote $\mathrm{sSet}_{/\mathrm{Sing}(X)}$ and $\mathbf{A}_X^o$ the bifibrant objects of $\mathbf{A}_X^o$. The bifibrant objects of sSet (with the usual model category structure) are Kan complexes. The map ψ now factors in the following way:

$$
\begin{tikzcd}
\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) \arrow[r, "\psi"] \arrow[d, "\text{if l.s.s.}"'] \arrow[dr, "\text{if c.}"] & p\mathrm{Shv}(X) \\
\mathrm{Shv}^A(X) \arrow[r, dashed] & \mathrm{Shv}(X) \arrow[u, dashed]
\end{tikzcd}
$$

If the A-stratification of X is conical, then the map ψ factors through $\mathrm{Shv}(X)$. If further X has locally singular shape, then this new map factors through $\mathrm{Shv}^A(X)$. Theorem A.9.3 of [Lur16] describes why this last map is an equivalence.

9 Symmetric monoidal infinity categories

Maximilien Péroux, 2017-11-06

Sources for this talk: [Gro15], [Lur09a].

9.1 Ordinary categories

The following is the main example to keep in mind.

Example 19. Let $(M, \otimes, \mathbf{1})$ be a monoidal category. Define a new category $M^{\otimes}$ with:

$$\begin{aligned} \text{objects:} &\quad (M_1, \dots, M_n) \text{ for } n \geqslant 0,\ M_i \in M, \\ \text{morphisms:} &\quad (M_1, \dots, M_n) \xrightarrow{\alpha, \{f_i\}} (L_1, \dots, L_k), \text{ where } \alpha : [k] \to [n] \text{ in } \Delta \text{ and } f_i : M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} \to L_i. \end{aligned}$$

This gives a functor which completely describes the monoidal structure of M, called the *Grothendieck opfibration*:

$$\begin{aligned} p \ : \ M^{\otimes} &\ \to \ \Delta^{op}, \\ (M_1, \dots, M_n) &\ \mapsto \ [n], \\ (\alpha, \{f_i\}) &\ \mapsto \ \alpha. \end{aligned}$$

Definition 34. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor. The object $\mathcal{C}_d$ is the fiber $d \in \mathcal{D}$ over p, defined as the pullback

$$\begin{tikzcd} \mathcal{C}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ * \arrow[r] & \mathcal{D} \end{tikzcd}$$

We would like the asignment $d \mapsto \mathcal{C}_d$ to be functorial, but it is not yet functorial.

Definition 35. Given a map $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$, it is a *p-coCartesian lift* of $\alpha : d_1 \to d_2$ in $\mathcal{D}$ if $p(f) = \alpha$ and

$$\begin{tikzcd} c_1 \arrow[r, "f"] \arrow[d, dash, "p"'] \arrow[rr, bend left, "\forall\, h"] & c_2 \arrow[r, dashed, "\exists ! g"] \arrow[d, dash, "p"] & c_3 \arrow[d, dash] \\ d_1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\forall\, \gamma"'] & d_2 \arrow[r, "\forall \beta"] & d_3 \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} \mathcal{C}_{f/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D}_{p(c_1)/} \arrow[d] \\ \mathcal{C}_{c_1/} \arrow[r] & \mathcal{C}_{p(c_1)/} \end{tikzcd}$$

commute.

Lemma 4. *Given $c \xrightarrow{f'} c'$ and $c \xrightarrow{f''} c''$ p-coCartesian lifts of α, then $c' \xrightarrow{\cong} c''$.*

Definition 36. A functor $p : \mathcal{C} \to \mathcal{D}$ is a Grothendieck opfibration if for all $c_1 \in \mathcal{C}$ and any morphism $\alpha \in \mathcal{D}$ with domain $p(c_1)$ there is a p-coCartesian lift $c_1 \to c_2$ of α. Given such $\alpha : p(c_1) = d_1 \to d_2$ in $\mathcal{D}$, we get a new functor

$$\begin{aligned} \alpha! \ : \ \mathcal{C}_{d_1} &\ \to \ \mathcal{C}_{d_2}, \\ c_1 &\ \mapsto \ c_2, \end{aligned}$$

where c_2 is the codomain of a lift of α.

Note that $\mathcal{D} \to \mathrm{Cat}$, given by $d \mapsto \mathcal{C}_d$ is still not functorial. This can be seen by considering $d_1 \xrightarrow{\alpha} d_2 \xrightarrow{\beta} d_3$. Going back to the main example $M^{\otimes}$, we have that the fiber $M^{\otimes}_{[n]}$ is canonically equivalent to $(M^{\otimes}_{[1]})^{\times n}$.

Definition 37. Consider the assignment

$$\iota_i = \begin{pmatrix} [1] & \to & [n], \\ 0 & \mapsto & i-1, \\ 1 & \mapsto & i, \end{pmatrix}^{op} \qquad \rightsquigarrow \qquad \begin{array}{rcl} (\iota_i)! \ : \ M^{\otimes}_{[n]} & \to & M^{\otimes}_{[1]} = M, \\ (M_1, \dots, M_n) & \mapsto & M_i. \end{array}$$

These induce new maps $\sigma = ((\iota_1)!, \dots, (\iota_n)!) : M^{\otimes}_{[n]} \times \cdots \times M^{\otimes}_{[1]}$, called *Segal maps.*

The *Segal conditions* are the conditions which, when satisfied, maked σ an equivalence. This approach can be generalized: given a Gorthendieck opfibration $\mathcal{C} \to \Delta^{op}$ respecting the Segal conditions, we get a monoidal category $M = \mathcal{C}_{[1]}$. Define $\otimes : M \times M \to M$ by letting $d_1 : [2] \to [1]$ be the face map, so then

$$\otimes \ : \ M \times M \xleftarrow[\sigma]{\cong} \mathcal{C}_{[2]} \xrightarrow{(d_1)!} \mathcal{C}_{[1]} = M.$$

The unit is defined by $[0] \to [1] \rightsquigarrow * \to \mathcal{C}_{[1]} = M$. To get a *symmetric* monoidal category, change the category Δ^{op} to Fin_*, the category of finite pointed sets.

9.2 The ∞-setup

Definition 38. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. A map $c_1 \xrightarrow{f} c_2$ is a *coCartesian lift* of $\alpha = p(f)$ in $\mathcal{D}$ if $\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$ is a trivial Kan fibration. A *coCartesian fibration* $p : \mathcal{C} \to \mathcal{D}$ is:

1. an inner fibration, and
2. for all $c_1 \in \mathcal{C}$, for all $\alpha = p(c_1) : d_1 \to d_2$ in $\mathcal{D}$, there is a p-coCartesian lift $c_1 \xrightarrow{f} c_2$.

Definition 39. A *monoidal ∞-category* is a coCartesian functor $p : M^{\otimes} \to N(\Delta^{op})$ that respects the Segal conditions.

This definition implies that $\mathrm{Ho}(M^{\otimes})$ inherits a monoidal structure.

Example 20. Here are two examples of monoidal ∞-categories.

1. Given M an ordinary monoidal category, $M^{\otimes} \xrightarrow{p} \Delta^{op}$, take the nerve $N(p) : N(M^{\otimes}) \xrightarrow{N(p)} N(\Delta^{op})$.
2. A nice model category gives an ∞-category via the coherent nerve N_{Δ}. That is, $N_{\Delta}(M_{CF}) \to N_{\Delta}(\Delta^{op}) = N(\Delta^{op})$.

9.3 Constructing an algebra

Given $M^{\otimes} \to \Delta^{op}$, a section $A : \Delta^{op} \to M^{\otimes}$ would maybe define an algebra. Consider the Segal map and face map:

$$\begin{array}{rcl} M^{\otimes}_{[n]} & \to & M \times \cdots \times M, \\ A_{[n]} & \to & (A^1_{[n]}, \dots, A^n_{[n]}), \end{array} \qquad \begin{array}{rcl} d_1 \ : \ [2] & \to & [1], \\ (A^1_{[2]}, A^2_{[2]}) & \to & A^1_{[1]}, \\ (A^1_{[2]}, A^2_{[2]}) & \to & A^1_{[2]} \otimes A^1_{[2]}. \end{array}$$

We would like $A^1_{[2]} \cong A^2_{[2]} \cong A^1_{[2]}$. The p-coCartesian lift would be $A^1_{[2]} \otimes A^1_{[2]} \to A^1_{[2]}$.

Definition 40. A morhpism $\alpha : [n] \to [k]$ in Δ is *convex* if it is a monomorphism and $\mathrm{im}(\alpha) = [\alpha(0), \alpha(n)]$ is an interval.

In particular, the ι_i are convex.

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10 Higher category theory in algebraic geometry

Micah Darrell, 2017-11-13

Sources for this talk: [BZFN10].

In [BZFN10], it is shown that for X, Y, Z *perfect stacks*, $QC(X \times_Z Y) \cong QC(X) \otimes_{QC(Z)} QC(Y)$. Then *Hochshild homology* may be restated as

$$HH(X) := QC(X) \otimes_{QC(X) \otimes QC(X)} QC(X) \cong QC(X \times_{X \times X} X) \cong QC(\mathrm{Map}(S^1, X)) = QC(\mathcal{L}X),$$

where $\mathcal{L}X$ is the *loop space* on X. Recall that we also have $\pi QC(X) = \mathcal{D}(X)$, the *derived category* of X.

10.1 Tensor of presentable ∞-categories

Definition 41. Pr^L is the sub-$(\infty, 1)$-category of presentable ∞-categories whose morphisms are colimit preserving functors.

The "L" is for "left-adjoint." If $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L$, then there is an ∞-category $\mathcal{C} \otimes \mathcal{D} \in \mathrm{Pr}^L$ with thte property that if $E \in \mathrm{Pr}^L$ and $F : \mathcal{C} \times \mathcal{D} \to E$ is a functor that preserves colimits separately in each variable, then F factors through $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$.

For *affine derived sheaves* A, we set $QC(A) = \mathrm{Mod}_A$. Then for a *derived stack* $X = \mathrm{colim}_{U \in \mathrm{Aff}_{/X}} U$, set $QC(X) = \lim_{U \in \mathrm{Aff}_{/X}} QC(U)$. Also note that $\pi : \infty cat \to CAT$ is a right adjoint to the nerve functor $N : CAT \to \infty cat$.

Definition 42. Let Z be a derived ring. An A-module M is called *perfect* if it is the smallest subcategory of Mod_A which contains A and is closed under finite colimits and retracts. The category of such modules is denoted $\mathrm{Perf}(A)$.

Then for a general *derived stack* X, an element $M \in QC(X)$ is called *perfect* is for any $F : U \to X$, for U affine, we have $f^*M \in \mathrm{Perf}(U)$. We call a derived stack a *perfect stack* if $QC(X) = \mathrm{IndPerf}(X)$, where Ind means take the inductive limits. Note that:

- $QC(X)$ has finite colimits, so if X is perfect, then $QC(X)$ is presentable.
- If X is a perfect stack, then $M \in \mathrm{Perf}(X) \iff (\mathrm{Hom}_{QC(X)}(M, -)$ commutes with colimits$)$.
- Quasi-compact schemes with affine diagonal are perfect stacks.

10.2 BZFN proof

To prove the given statement atthe beginning, the authors first show that it holds for compact objects (equivalently, perfect objects). They use the exterior product

$$\begin{array}{rccc} \boxtimes \;:\; & QC(X)^c \times QC(X)^c & \xrightarrow{\sim} & QC(X \times Y)^c, \\ & (M, N) & \mapsto & P_1^* M \otimes P_2^* N. \end{array}$$

To prove the general case, we use a corollary of the *adjoint functor theorem*, which says $(\mathrm{Pr}^L)^{op} \cong \mathrm{Pr}^R$. We know that $QC(X \times Y) \cong QC(X) \otimes QC(Y)$ in the regular category setting. Begin with

$$X \times_Z Y \xrightarrow{\pi} X \times Y \rightrightarrows X \times Z \times Y \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ X \times Z \times Z \times Y \ \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

then apply QC, which is a contravariant functor, to get

$$QC(X \times_Z Y) \xleftarrow{\pi^*} QC(X \times Y) \leftleftarrows QC(X \times Z \times Y) \ \substack{\leftarrow\\\leftarrow\\\leftarrow} \ QC(X \times Z \times Z \times Y) \ \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} \cdots.$$

The general term is $QC(X) \otimes QC(Z) \otimes \cdots \otimes QC(Z) \otimes QC(Y)$. Note that this is just the 2-sided *bar construction.*

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11 Spectra

Haldun Özgür Bayındır, 2017-11-20

Sources for this talk: [Gro15] Section 5.2

11.1 Reviewing spectra

Let E be a *prespectrum*, with $E_n \in \mathrm{sSet}_*$ and maps $\sigma_n : S^1 \wedge E_n \to E_{n+1}$. Recall the *smash product* is given by $S^1 \wedge E_n = S^1 \times E_n / S^1 \vee E_n$. There suspension and loop space are adjoints

$$(\Sigma, \Omega) \ : \ \mathrm{sSet}_* \rightleftarrows \mathrm{sSet}_*.$$

Definition 43. A prespectrum E is a *spectrum* if the adjoint maps $E_n \to \Omega E_{n+1}$ are homeomorphisms.

We now have a functor

$$\begin{tikzcd}
\Sigma^\infty \ : \ \mathrm{sSet}_* \arrow[rr] \arrow[dr] & & \mathrm{Sp}, \\
X \arrow[dr, mapsto] & \mathrm{preSp} \arrow[ur] & E_n \ = \ \mathrm{colim}_{\sigma_i} \Omega^i \Sigma^i E'_n \\
& E'_n \arrow[ur, mapsto] \arrow[d, equal] & \\
& \Sigma^n X &
\end{tikzcd}$$

where Sp is the category of spectra. This induces a smash product on the homotopy category Ho(Sp), as well as on Sp.

Remark 7. There are new models of spectra, which are all equivalent: LMS (Lewis–May–Steinberger), EKMM (Elmendorf–Kriz–Mandell–May), symmetric spectra (Hovey–Shipley–Smith), MMSS (Mandell–May–Schwede–Shipley).

To get a (commutative) ring spectrum, we would like the map $E \wedge E \to E$ to be unital and associative (and commutative).

Remark 8. Spectra are very useful. For E, F (commutative) ring spectra, we have

- $E_\bullet(-)$ is a homology theory,
- $E_\bullet F = \pi_\bullet E \wedge F$ is a (commutative) ring with power operations,
- $E^\bullet(-)$ is a cohomology theory,
- $E^n F = \pi_{-n} \mathrm{Map}_{\mathrm{Sp}}(F, E)$ incorprates the mapping space, and
- $E^n X = \pi_{-n} \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X_*, E)$ are (commutative) rings, for $X \in \mathrm{sSet}$.

We also get algebra structures and power operations on spectral sequences that allow us to calculate (for example) THH and $E_\bullet F$.

11.2 Spectra in ∞-categories

All ∞-categories will be presentable.

Definition 44. Let X, Y, Z be objects in a category $\mathcal{C}$. A *triangle* in $\mathcal{C}$ is a commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, "h" description] \arrow[r, phantom, shift right=4, "\Nearrow"] & Y \arrow[d, "g"] \\
0 \arrow[r] \arrow[ur, phantom, shift left=4, "\Swarrow"] & Z,
\end{tikzcd}$$

with $g \circ f \simeq 0$ and $h \simeq 0$. If the square is a pullback, then the triangle is *exact* (forming the category $\mathcal{C}^{\Omega}$), and if it is a pushout, then the triangle is *coexact* (forming the category $\mathcal{C}^{\Sigma}$).

Example 21. For example, let $X = S^1$ the circle and $Y = D^2$ the disk. Then we have $Z = S^2$, giving the coexact triangle

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ * \arrow[r] & S^2. \end{tikzcd}$$

Let $\mathcal{C}$ be a finitely (co)complete pointed ∞-category. Then we hve evaluation maps

$$\begin{array}{rcccc} \mathrm{ev}_{(0,0)} & : & \mathcal{C}^{\Sigma} & \to & \mathcal{C}, \\ \mathrm{ev}_{(1,1)} & : & \mathcal{C}^{\Omega} & \to & \mathcal{C}, \end{array}$$

both of which are acyclic Kan fibrations. Using these we can try to define suspensions with the diagram

$$\begin{tikzcd} \begin{pmatrix} \text{contractible} \\ \text{object} \end{pmatrix} \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{C}, \mathcal{C}^{\Sigma}) \arrow[d, "\mathrm{ev}_{(0,0)}"] \\ \Delta^0 \arrow[r, "\mathrm{id}_{\mathcal{C}}"] & \mathrm{Map}(\mathcal{C}, \mathcal{C}). \end{tikzcd}$$

We then get maps $\Sigma : \mathcal{C} \to \mathcal{C}^2 \to \mathcal{C}$ and $\Omega : \mathcal{C} \to \mathcal{C}^{\Omega} \to \mathcal{C}$, giving an adjunction $(\Sigma, \Omega) : \mathcal{C} \leftrightarrows \mathcal{C}$.

Definition 45. A pointed, (co)complete category $\mathcal{C}$ is *stable* if every triangle is exact iff it is coexact.

Theorem 8. *The following are equivlent:*

1. (Σ, Ω) *is an equivalence of categories.*
2. $\mathcal{C}$ *is stable.*
3. *Every commutative squares is a pullback iff it is a pushout.*

To define a *spectrum*, we begin with a functor $X : N(\mathbf{Z} \times \mathbf{Z}) \to \mathcal{C}$ such that $X(i, j) = 0$ for all $i \neq j$. This gives a commutative diagram.

$$\begin{tikzcd} & & \iddots \\ & 0 \arrow[r] & X_{n+1} \\ 0 \arrow[r] & X_n \arrow[r] \arrow[u] & 0 \arrow[u] \\ X_{n-1} \arrow[r] \arrow[u] & 0 \arrow[u] & \\ \iddots & & \end{tikzcd}$$

where $X_n = X(n, n)$. We also have maps $\alpha_{n-1} : \Sigma X_{n-1} \to X_n$ and $\beta_n : X_n \to \Omega X_{n+1}$. Then we say that X is a *spectrum below* n if β_m is a weak equivalence for all $m < n$, and simply a *spectrum* if β_n is a weak equivalence for all n.

The next step is to define a functor $\mathrm{preSp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$, which can be seen in Proposition 5.23 of [Gro15].

12 Topoi and ∞-topoi

Adam Pratt, 2017-11-27

Sources for this talk: [JT84], [MLM94], [BS10], [Cis16], [Lur09a].

12.1 Elementary topoi

The view to keep is that elementary topoi generalize sets and ∞-topoi generalize spaces. In topological spaces, one can think in terms of *locales* and *frames*. Every topological space has a frame of open sets $\mathrm{Op}(X)$.

Definition 46. A *frame* is a poset that has all small *joins* ($\vee$) and finite *meets* ($\wedge$) such that $X \wedge (\bigvee_i Y_i) = \bigvee_i (X \wedge Y_i)$.

We can think of $\vee$ as $\cup$ and $\wedge$ as $\cap$, though they are not exactly the same. Note that the category of locales is the opposite of the category of frames.

Remark 9. The category Set of sets has the following properties:

1. Has finite limits and all colimits
2. Is cartesian closed (has product, exponential object X^Y, and terminal object $\emptyset$)
3. Has a subobject classifier $\subseteq$

Definition 47. An *elementary topos* is a category which has the following properties:

1. Has finite limits
2. Is cartesian closed
3. Has a subobject classifier

Example 22. Set is an elementary topos. A *Heyting algebra* is also an elementary topos - this is a poset that has the following properties:

1. Products are meets
2. Is cartesian closed
3. Subobject classifier is $\implies$, with $((c \wedge a) \subseteq b) \iff ((c \subseteq a) \implies b)$

A Heyting algebra is the weakest setting to do logic (that is, has modus ponens, or $((P \implies Q) \wedge P) \implies Q$). Infact, a frame is equivalent to a complete Heyting algebra, as it has all small joins.

12.2 Sheaves

Recall that a *presheaf* of sets on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$, and the category of presheaves is $\mathrm{preShv}(\mathcal{C}) = [\mathcal{C}^{op}, \mathrm{Set}]$.

Example 23. $\mathrm{preShv}(\mathcal{C})$ is an elementary topos.

Definition 48. A *site* $(\mathcal{C}, J)$ is a small category $\mathcal{C}$ equipped with a *Grothendieck topology* J. A *Grothendieck topos* is a category equivalent to the ctegory of sheaves on some site.

Example 24. Set is a Grothendieck topos. It is a category of sheaves on a point.

Note that Set is terminal in ShTopos, the 2-category of topoi. The unique morphism is the global sections functor.

Theorem 9. *A Grothendieck topos is an elementary topos that is comploete (has all small colimits) and has a small generating set.*

Remark 10. Note that:

- Set is cocomplete and any nonempty set generates Set.
- Any scheme can be seen as a (locally ringed) Grothendieck topos.
- The locale axioms are the Grothendieck topos in a 0-category sense. Moreover, any locale is a $(0,1)$-Grothendieck topos and a Heyting algebra is a $(0,1)$-elementary topos.

12.3 The ∞ setting

Now we move from sets to spaces. Recall that ∞Grpd, the category of ∞-groupoids, is presentable, colimits are universal, and has small "self-reflection" (see [Lur09a] for more on this).

Definition 49. [Giraud–Rezk–Lurie]
An $(\infty,1)$-*topos* is a presentable $(\infty,1)$-category with universal colimits and an *object classifier*, which is an object c such that $c \mapsto \mathrm{Core}(\mathcal{C}_{/c})$ is a representable ∞-functor $\mathcal{C}^{op} \to \infty\mathrm{Grpd}$.

Example 25. ∞Grpd is terminal, with the unique morphism being the global sections functor.

These objects also appear in *spectral algebraic geometry*, which studies spectral schemes and schemes built from E_∞ rings. Note that structure sheaves on E_∞ give us ∞-topoi. Moreover, elementary ∞-topoi give categorical semantics for ∞-topoi.

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13 K-theory

Jack Hafer, 2017-12-04

Sources for this talk: [Wei13] Sections II.9 and IV.8, [BGT13].

The goal of this talk will be to understand the statement "$K : \mathrm{Cat}_\infty^{perf} \to S_\infty$ is additive invariant," where the superscript $perf$ means idempotent-complete categories. The statement may also be viewed as saying that K inverts Morita eqivalences, preserves filtered colimits and satisfies Waldhausen additivity. A more ambitious goal would be to show that this is a universal such functor.

13.1 Waldhausen categories

Definition 50. A *category with cofibrations* (cofibration indicated by $\rightarrowtail$) is a category satisfying:

1. every isomorphism is a cofibration,
2. there exists a 0-object with $0 \rightarrowtail A$, and
3. if $A \rightarrowtail A$ and $AtoC$, then $B \cup_A C$ exists and $C \rightarrowtail B \cup_A C$.

Here $B \cup_A C$ indicates the pushout of B and C with respect to A. In a category with cofibrations we then have that $B \sqcup C = B \cup_0 C$ and $B/A = B \cup_A 0$. Reall a *cofiber sequence* is a sequence $A \rightarrowtail B \twoheadrightarrow C$.

Definition 51. A *category with weak equivalences* (weak equivalence indicated by $\xrightarrow{\sim}$) is a category satisfying:

1. every isomorphism is a cofibration,
2. there exists a 0-object with $0 \rightarrowtail A$, and
3. for every commuting diagram of the form

$$\begin{tikzcd} C \arrow[d, "\wr"] & A \arrow[l] \arrow[r, tail] \arrow[d, "\wr"] & B \arrow[d, "\wr"] \\ C' & A' \arrow[l] \arrow[r, tail] & B', \end{tikzcd}$$

we have $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$.

Definition 52. A *Waldhausen category* is a category with cofibrations and weak equivalences.

We write $K_0(\mathcal{C})$, for $\mathcal{C}$ Waldhausen, to denote the abelian group with generators $[C]$ for every $C \in \mathrm{obj}(\mathcal{C})$, and relations $[C] = [C']$ for every $C \xrightarrow{\sim} C'$. For every *cofiber sequence* $B \rightarrowtail C \twoheadrightarrow B/C$, we also have $[C] = [B] + [B/C]$. From the properties of a Waldhausen category, we immediately have that

- $[0] = 0$,
- $[C] = 0$ if $C \xrightarrow{\sim} 0$,
- $[B \cup_A C] = [B] + [C] - [A]$.

Recall that a functor is *exact* if it preserves the zero object, cofibrations, weak equivalences and pushouts.

Definition 53. A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is such that

- $\mathcal{A} \subseteq \mathcal{C}$ is exact,
- cofibrations in $\mathcal{A}$ are maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ whose cokernel lies in $\mathcal{A}$,
- weak equivalences in $\mathcal{A}$ are weak equivalences in $\mathcal{C}$.

Now define a sequence of Waldhausen categories, for $\mathcal{C}$ a category with cofibrations, in the following way:

$S_0\mathcal{C} = 0$,

$S_1\mathcal{C} = \mathcal{C}$, objects $0 \rightarrowtail A$,

$S_2\mathcal{C}$ has objects cofiber sequences $E : A_1 \rightarrowtail A_2 \twoheadrightarrow A_{12}$ and morphisms

$$\begin{tikzcd} E \arrow[d, "f"] & : & A_1 \arrow[r, tail] \arrow[d, "u_1"] & A_2 \arrow[r, two heads] \arrow[d, "u_2"] & A_{12} \arrow[d, "u_{12}"] \\ E' & : & A'_1 \arrow[r, tail] & A'_2 \arrow[r, two heads] & A'_{12}. \end{tikzcd}$$

A map $f \in S_2\mathcal{C}$ is a cofibration if u_1, u_{12} are cofibrations and $A'_1 \cup_{A_1} A_2 \to A'_2$ is a cofibration. A map f is a weak equivalence if all of u_1, u_2, u_{12} are weak equivalences.

$S_n\mathcal{C}$ has objects $A_\bullet : 0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$ together with a choice of quotients A_i/A_{i-1}. The following diagram (Δ) commutes:

$$\begin{tikzcd} & & & \cdots \arrow[r, tail] & A_{nn} \\ & & & & \vdots \arrow[u, two heads] \\ & & A_{23} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{2n} \arrow[u, two heads] \\ & A_{12} \arrow[r, tail] & A_{13} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{1n} \arrow[u, two heads] \\ A_1 \arrow[r, tail] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_n \arrow[u, two heads] \end{tikzcd}$$

Morphisms of $S_n\mathcal{C}$ are natural tranformations of cofiber sequences.

Note that $S_n\mathcal{C}$ is Waldhausen.

13.2 Simplicial Waldhausen categories

The payoff from this long construction is that we get a *simplicial* Waldhausen category. We get $\partial_0 : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by deleting the bottom row in (Δ). That is,

$$\partial_0(A_\bullet) = (0 = A_{11} \rightarrowtail A_{12} \rightarrowtail A_{13} \rightarrowtail \cdots \rightarrowtail A_{1n})$$

together with a choice of quotients. We also have $\partial_0(A_\bullet)_{ij} = A_{i+1,j+1}$. Note that ∂_0 is exact. For $0 < i \leqslant n$, we have $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by omitting A_{i*} and the column containing A_i in (Δ). Dually, we get $S_i : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ by duplicating the column that contains A_i and reindexing.

Remark 11. This shows that $S_\bullet\mathcal{C}$ is a simplicial Waldhausen category and the full subcategory $wS_\bullet\mathcal{C}$ of weak equivalences is a simplicial Waldhausen subcategory of $S_\bullet\mathcal{C}$.

The geometric realization $|wS_\bullet\mathcal{C}|$ is a connected simplicial space. It may be shown that $\pi_1|wS_\bullet\mathcal{C}| = K_0(\mathcal{C})$, and the Σ-Ω (suspension-loop space) adjunction gives that $K_i(\mathcal{C}) = \pi_{i+1}|wS_\bullet\mathcal{C}|$.

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