ESP Math 182

Worksheet 17

Spring 2018

13 March 2018

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1. **Warm up:** The following series are all convergent. Indicate which test applies to determine convergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/3}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n-2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{-n}}{n\ln(n)}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)\cos(\pi n)}{n}$

2. Use l'Hopital's rule to evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

   (b) $\displaystyle\lim_{x\to 0^+} x^{\sin(x)}$

   (c) $\displaystyle\lim_{y\to 0} \frac{\tan(y) - y}{y^3}$

   (d) $\displaystyle\lim_{z\to\infty} ze^{1/z} - z$

3. Recall the integral test says that for a non-increasing function f, the sum $\sum_{n=N}^{\infty} f(n)$ converges if and only if the integral $\int_N^{\infty} f(x)\,dx$ is finite.

   (a) Show that $\displaystyle\sum_{n=2}^{\infty} \left(\frac{e}{n}\right)^n$ converges.

   (b) Show that $\displaystyle\int_1^{\infty} \frac{e^y}{y^y}\,dy$ converges.

   (c) Determine whether or not $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$ converges.

4. You are given the two inequalities

$$x - \frac{x^2}{2} < \ln(1+x) < x, \qquad \text{for } x > 0, \qquad (1)$$

$$\ln(1) + \ln(2) + \cdots + \ln(n-1) < \int_1^n \ln(x)\ dx < \ln(2) + \cdots + \ln(n), \qquad \text{for } n \geqslant 2. \qquad (2)$$

Use them to answer the following questions.

(a) Use inequality (1) to prove that $\lim_{x \to 0^+} \left[\frac{\ln(1+x)}{x} \right] = 1$.

(b) Use part (a) to prove that $\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)^n \right] = e$.

(c) Show that $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ converges for $0 < a < e$ and diverges for $a > e$.

(d) Use inequality (2) to show that $\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$.

(e) Use part (d) to determine if the series from part (c) converges if $a = e$.