Stat401 9/1/17

HW sec 1.2 due Wed. 

## Sec 1.3 - Probability Set Function

Motivation:

Given an experiment, let $\mathcal{C}$ be the sample space of all possible outcomes.
We want to assign probabilities to events, i.e. subsets of $\mathcal{C}$

ex) 6-sided die

$$\mathcal{C} = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 2, 3\} \subset \mathcal{C}$$

$$B = \{2, 6\} \subset \mathcal{C}.$$

Goal: Find the chance these events occur.

def) Probability

Let $\mathcal{C}$ be a sample space.
Let $\mathcal{B}$ be the set of events.
We say $\mathbb{P}$ is a real-valued set function defined on $\mathcal{B}$ and is called a probability set function if

1. $\mathbb{P}(C) > 0$ for all $C \in \mathcal{B}$
2. $\mathbb{P}(\mathcal{C}) = 1$
3. If $\{C_n\}$ is a sequence of events in $\mathcal{B}$ and

$C_m \cap C_n = \emptyset$ for all $m \neq n$
then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n)$$

## Remarks and Interpretations

1. How can we think about $\mathcal{B}$?

ex) Toss 2 coins.

$$\mathcal{C} = \{(H,H), (H,T), (T,H), (T,T)\}.$$

Possible Events:

$\{(H,H), (H,T)\}$.

$\{(T,H), (T,T)\}$

$\{(H,H), (H,T), (T,H)\}$.

$\{(H,H), (H,T), (T,H), (T,T)\}$.

$\vdots$

This list of ALL events in $\mathcal{B}$

2) what does

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n)$$

for $\underbrace{C_m \cap C_n = \emptyset \ \forall m \neq n}_{\text{events are pairwise disjoint!}}$.

mean?

events are pairwise disjoint!

def) pairwise disjoint.
- every 2 different sets in the family are disjoint / no overlap.

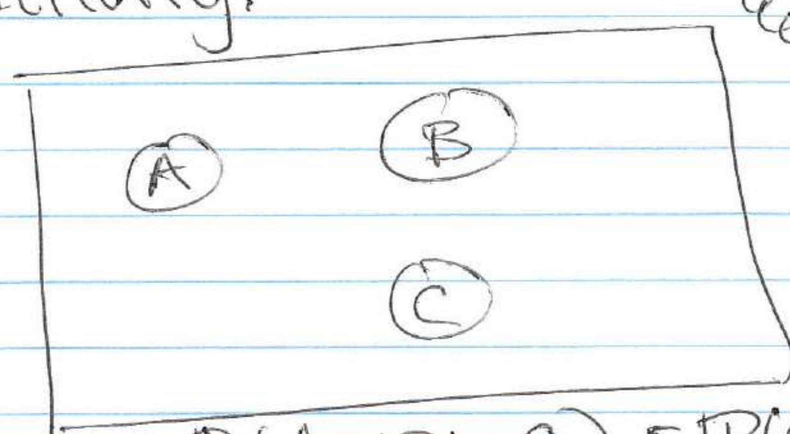
ex) $\mathcal{C} = \{1, 2, 3, 4\}$

Only events: $\{1\}, \{2\}, \{3, 4\}$.

we end up with countable additivity.

$\mathcal{C}$

ex)




$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$

word often dropped.

(3) Prob set funct. tells us how
the prob. is distributed over
the events $\mathcal{B}$.

$\mathbb{P}$ = prob. function.

we can think of $\mathbb{P}(C)$ = probability
that the event $C$ occurs.

(4) $\{C_n\}$. is said to be a
$\underbrace{\text{mutually exhaustive}}$ collection

union of events is $\mathcal{C}$.

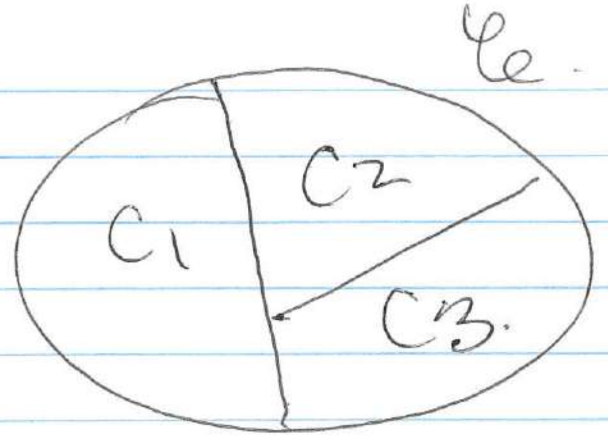
$$\bigcup_{n=1}^{\infty} C_n = \mathcal{C}.$$

$$\text{so} \quad \mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n) = 1$$

disjoint.

→ A mutually exclusive and
exhaustive collection of events
is a partition of $\mathcal{C}$.

ex).

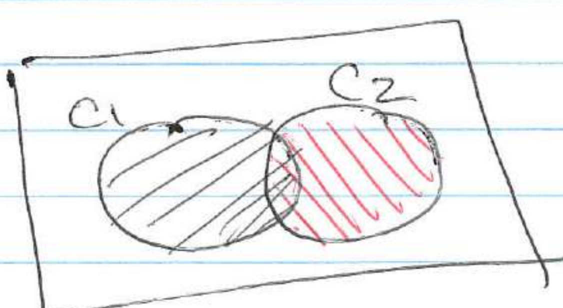



$C_1 \cup C_2 \cup C_3 = \mathcal{C}$.

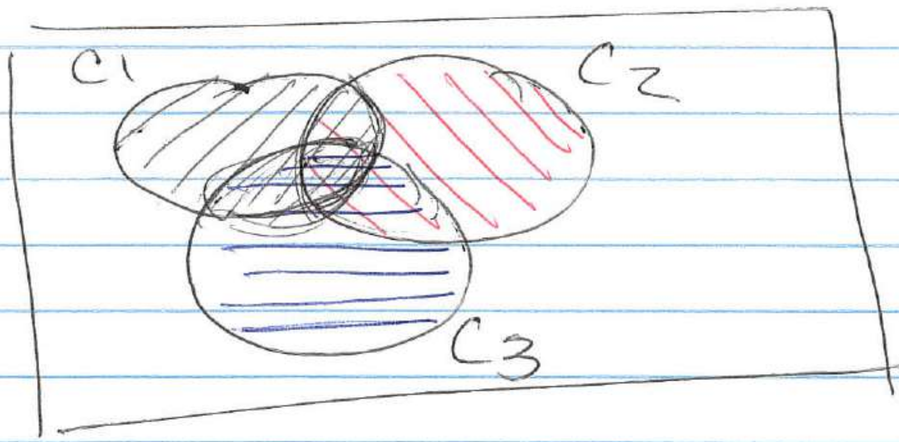
Properties of Probability Functions

1. Complement Rule $\mathbb{P}(C) = 1 - \mathbb{P}(C^c)$
2. $\mathbb{P}(\emptyset) = 0$
3. If $C_1 \subset C_2$ then $\mathbb{P}(C_1) \leq \mathbb{P}(C_2)$
4. $0 \leq \mathbb{P}(C) \leq 1$
5. Addition Rule

$$\mathbb{P}(C_1 \cup C_2) = \mathbb{P}(C_1) + \mathbb{P}(C_2) - \mathbb{P}(C_1 \cap C_2)$$




$$\mathbb{P}(C_1 \cup C_2 \cup C_3) = \mathbb{P}(C_1) + \mathbb{P}(C_2) + \mathbb{P}(C_2) - \mathbb{P}(C_1 \cap C_2) - \mathbb{P}(C_2 \cap C_3) - \mathbb{P}(C_1 \cap C_3) + \mathbb{P}(C_1 \cap C_2 \cap C_3)$$




$\mathbb{P}(\mathcal{C}) = 1$

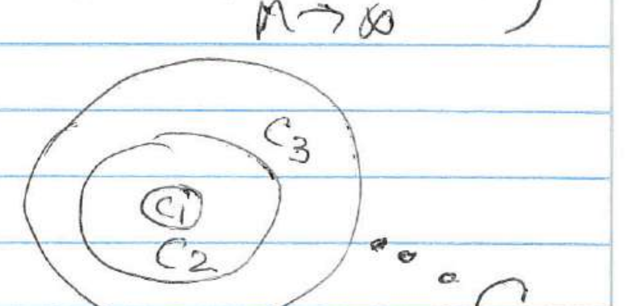
(6) If $\{C_n\}$ is a monotone increasing sequence then

$$\lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \mathbb{P}\left(\lim_{n\to\infty} C_n\right)$$

If $\{C_n\}$ is a monotone decreasing sequence then

$$\lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} C_n\right) = \mathbb{P}\left(\lim_{n\to\infty} C_n\right)$$

Proof. Suppose $C_n \uparrow C$



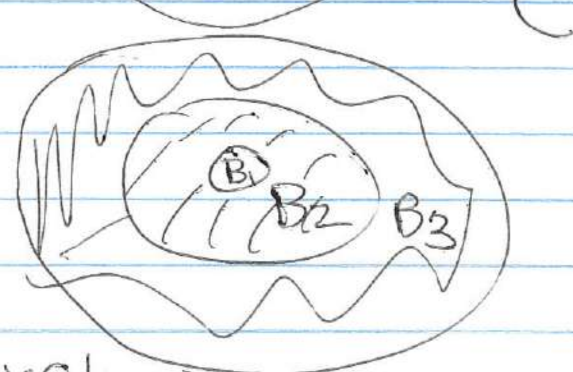

Let $B_1 = C_1$

$B_2 = C_2 \setminus C_1$

$B_3 = C_3 \setminus C_2$

$\vdots$

$B_n = C_n \setminus C_{n-1}$




$B_n$ is mutually exclusive.

Also $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n$

$$\mathbb{P}\left(\lim_{n\to\infty} C_n\right) = \mathbb{P}(C)$$
$$= \mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right)$$
$$= \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right)$$
$$= \sum_{n=1}^{\infty} \mathbb{P}(B_n) \text{ ; disjoint.}$$

= continued next time.