Stat 401 9/11/17
Sec 1.4: 8, 10, 25, 34 } due 9/18
Sec 1.5: 3, 6, 8

Last time: Sec 1.4.

Multiplication Rule

$$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(C_1) \cdot \mathbb{P}(C_2 \mid C_1) \quad \text{(2 events)}$$

$$\mathbb{P}(C_1 \cap C_2 \cap \cdots \cap C_n) = \mathbb{P}(C_1)\mathbb{P}(C_2 \mid C_1)\mathbb{P}(C_3 \mid C_1 \cap C_2) \cdot \ldots \cdot \mathbb{P}(C_n \mid C_1 \cap C_2 \cap \cdots \cap C_{n-1})$$
(n events)

Proof for $n=3$ (Proof for general n by induction)

Show $\mathbb{P}(C_1 \cap C_2 \cap C_3) = \mathbb{P}(C_1)\mathbb{P}(C_2 \mid C_1)\mathbb{P}(C_3 \mid C_1 \cap C_2)$

$$\text{LHS} = \mathbb{P}((C_1 \cap C_2) \cap C_3)$$
$$= \mathbb{P}(C_1 \cap C_2) \cdot \mathbb{P}(C_3 \mid C_1 \cap C_2)$$
def for 2 events.
$$= \mathbb{P}(C_1) \cdot \mathbb{P}(C_2 \mid C_1) \cdot \mathbb{P}(C_3 \mid C_1 \cap C_2)$$
$\square$

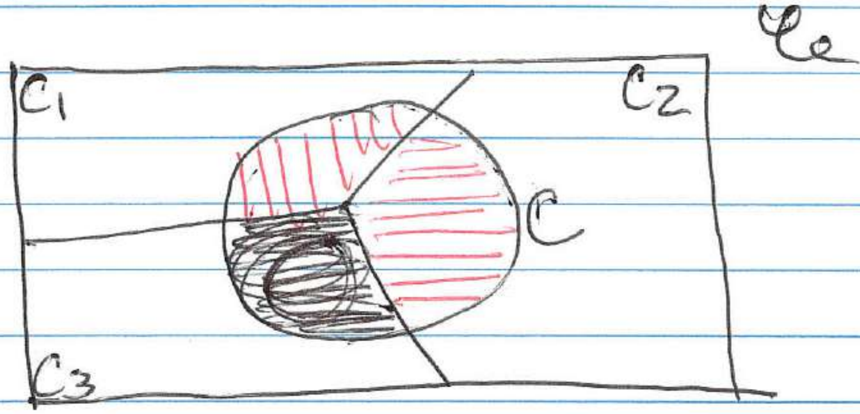
def | Law of Total Probability (LOTP)

$$\mathbb{P}(C) = \sum_{i=1}^{k} \mathbb{P}(C_i) \cdot \mathbb{P}(C \mid C_i) \text{ where}$$

$C_1, C_2, \ldots, C_k$ form a partition of $\mathcal{C}$ and C is another event.

ex) $k=3 \quad \mathcal{C} = C_1 \cup C_2 \cup C_3$

Take an event C



$$\mathbb{P}(C) = \mathbb{P}(C_1 \cap C) + \mathbb{P}(C_2 \cap C) + \mathbb{P}(C_3 \cap C)$$

$$= \mathbb{P}(C_1)\mathbb{P}(C|C_1)$$

$$+ \mathbb{P}(C_2)\,\mathbb{P}(C|C_2)$$

$$+ \mathbb{P}(C_3)\cdot\mathbb{P}(C|C_3)$$

$$= \sum_{i=1}^{3} \mathbb{P}(C_i)\cdot\mathbb{P}(C|C_i)$$

<u>Thm</u>) Bayes' Thm

mult Rule

$$\mathbb{P}(C_i|C) = \frac{\mathbb{P}(C \cap C_i)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C_i)\mathbb{P}(C|C_i)}{\sum_{j=1}^{k} \mathbb{P}(C_j)\mathbb{P}(C|C_j)}$$

LOTP

Ex 4

Bowl 1: 3 red 7 blue

Bowl 2: 8 red 2 blue.

Roll die. 5 or 6 select Bowl 1.

Person draws blue chip

Find prob chip came from bowl I?

SOL Define events.

$$C_1 = \text{Bowl } 1$$

$$C_2 = \text{Bowl } 2$$

$$C = \text{blue chip.}$$

Find $\mathbb{P}(C_1 \mid C)$.

Info: $\mathbb{P}(C_1) = \frac{2}{6} = \frac{1}{3}$

$\mathbb{P}(C_2) = \frac{4}{6} = \frac{2}{3}$

$\mathbb{P}(C \mid C_1) = \frac{7}{10}$ $\quad \mathbb{P}(C \mid C_2) = \frac{2}{10}$

$$\mathbb{P}(C_1 | C) = \frac{\mathbb{P}(C_1 \cap C)}{\mathbb{P}(C)}$$

$$= \frac{\mathbb{P}(C_1)\mathbb{P}(C | C_1)}{\mathbb{P}(C_1 \cap C) + \mathbb{P}(C_2 \cap C)}$$

Bayes' Thm.

$$= \frac{\mathbb{P}(C_1)\mathbb{P}(C | C_1)}{\mathbb{P}(C_1)\mathbb{P}(C|C_1) + \mathbb{P}(C_2)\mathbb{P}(C|C_2)}$$

$$= \frac{\left(\frac{2}{6}\right)\left(\frac{7}{10}\right)}{\left(\frac{2}{6}\right)\left(\frac{7}{10}\right) + \left(\frac{4}{6}\right)\left(\frac{2}{10}\right)}$$

$$= \frac{7}{11}$$

(b) What is prob selected from Bowl 2, given that I have a blue chip?

$$\mathbb{P}(C_2 | C) = 1 - \mathbb{P}(C_2^c | C)$$

$$= 1 - \mathbb{P}(C_1 | C)$$

$$= 1 - \frac{7}{11}$$

$$= \frac{4}{11}$$

Ex 5)

machine made by

$$\mathbb{P}(C_1) = 0.3$$
$$\mathbb{P}(C_2) = 0.45$$
$$\mathbb{P}(C_3) = 0.25$$

C = defective.

$$\mathbb{P}(C \mid C_1) = 0.02 \quad \mathbb{P}(C \mid C_2) = 0.03$$
$$\mathbb{P}(C \mid C_3) = 0.02$$

a) Find $\mathbb{P}(C)$.

$$\mathbb{P}(C) \overset{\text{LOTP}}{=} \sum_{i=1}^{3} \mathbb{P}(C_i)\,\mathbb{P}(C \mid C_i)$$

$$= (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02)$$

$$= 0.0245$$

b) $$\mathbb{P}(C_3 \mid C) = \frac{\mathbb{P}(C_3)\,\mathbb{P}(C \mid C_3)}{\mathbb{P}(C)}$$

$$= \frac{(0.25)(0.02)}{0.0245} = 0.2041$$

Remarks)

- Call $\mathbb{P}(C_1)$, $\mathbb{P}(C_2)$, etc. as prior probabilities of $C_1, C_2, \ldots$

- call $\mathbb{P}(C_1|C)$ and $\mathbb{P}(C_2|C)\ldots$ as posterior probabilities

Bayesian Inference.

$$\text{posterior} = \text{prior} \times \frac{\text{likelihood}}{\text{marginal likelihood}}$$

$$\mathbb{P}(H|E) = \mathbb{P}(H) \times \frac{\mathbb{P}(E|H)}{\mathbb{P}(E)}.$$

H = hypothesis

E = event of interest.

Independence

def) Independence (2 Events).

Let C_1, C_2 be 2 events.
C_1 and C_2 are independent if and only iff

$$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(C_1) \cdot \mathbb{P}(C_2).$$

→ events don't depend upon each other.

Proof $\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(C_1)\,\mathbb{P}(C_2 \mid C_1)$

$= \mathbb{P}(C_1) \cdot \mathbb{P}(C_2)$.

Note: If C_1 and C_2 indep. then

C_1 and C_2^c are indep,
C_1^c and C_2 are indep,
C_1^c and C_2^c are indep.

def Pairwise Independent.

For $n \geq 3$, pairwise indep indicates that

$$\mathbb{P}(C_i \cap C_j) = \mathbb{P}(C_i)\,\mathbb{P}(C_j), \text{ for all } i \neq j$$

def Mutually Independent.

The n events $C_1, C_2, \ldots, C_n$ are mutually indep if and only if for every collection of k of these events $2 \leq k \leq n$, the following is true:

If $d_1, d_2, \ldots, d_k$ are k distinct integers from $1, 2, \ldots, n$, then

$$\mathbb{P}(C_{d_1} \cap C_{d_2} \cap \cdots \cap C_{d_k})$$
$$= \mathbb{P}(C_{d_1})\,\mathbb{P}(C_{d_2}) \cdot \ldots \cdot \mathbb{P}(C_{d_k}).$$

Note: Mutually indep $\Rightarrow$ pairwise indep.
pairwise indep $\not\Rightarrow$ mutually indep.

see Example 1.4.9 in text for counterexample.

ex| C_1, C_2, C_3, C_4, C_5 are events.

pairwise indep. C_1 + C_2 indep.

$k = 2$

C_1 + C_2	indep.
C_1 + C_3	"
C_1 + C_4	"
C_1 + C_5	"
C_2 + C_3	"
⋮	

$k = 3$

C_1 + C_2 + C_3	indep.
C_1 + C_2 + C_4	"
C_1 + C_2 + C_5	"
C_2 + C_3 + C_4	"
⋮	

$k=4$ $\quad C_1 + C_2 + C_3 + C_4$ indep.

$\quad\quad\quad\; C_1 + C_2 + C_3 + C_5$ indep.

$\quad\quad\quad\quad\quad \vdots$

$k=5$ $\quad C_1 + C_2 + C_3 + C_4 + C_5$ indep.

mutually indep —

all of these true @ the same time.