hw: Last Name, First Name.

Sec 1.5 continued (RVs)

def) Cumulative distribution Function (CDF)

- For a RV X (discrete or continuous), the CDF $F_X(x)$ is defined by (X ← RV)

$$F_X(x) = P(\{c \in \mathcal{C} : X(c) \le x\})$$
$$= P_X((-\infty, x])$$
$$= P(X \le x)$$

Purpose: It accumulates the probabilities less than or equal to x.

ex) Discrete RV X

Roll a die.

X = outcome $= \{1, 2, 3, 4, 5, 6\}$

Space/ Range value x	1	2	3	4	5	6
pmf → $p_X(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

CDF of X:

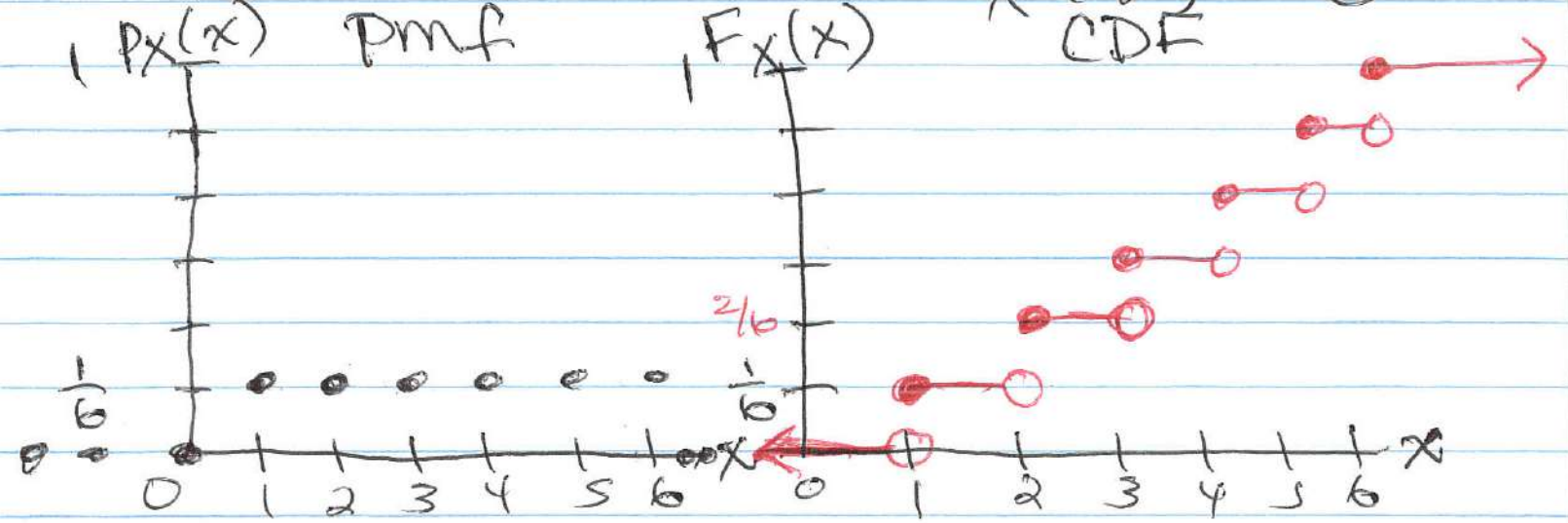
$$F_X(x) = \sum_{t \le x} p_X(t)$$

$$= \begin{cases} 0, & x < 1 \\ \frac{1}{6}, & 1 \le x < 2 \\ \frac{2}{6}, & 2 \le x < 3 \\ 3/6, & 3 \le x < 4 \\ 4/6, & 4 \le x < 5 \\ 5/6, & 5 \le x < 6 \\ 1, & 6 \le x \end{cases}$$

cover all real #s

max value of $F_X(x) = 1$

min value of $F_X(x) = 0$



$$F_X(x) = \mathbb{P}(X \le x)$$

ex] Continuous RV X

Suppose X is a number chosen at random between 0 and 1.

Take $\mathbb{P}_X((a,b)) = b-a$, $0<a<b<1$

Find the CDF of X.

Sol)

Case 1: If $x \le 0$, $\mathbb{P}(X \le 0) = 0$

Case 2: If $x \ge 1$, $\mathbb{P}(X \le 1) = 1$

1, 2, 3, 4, 5, ...

$\mathbb{P}(X \le 2) = 1$

Case 3: If $0 < x < 1$

$$\mathbb{P}(X \le x) = \mathbb{P}(0 < X \le x) = \mathbb{P}_X((0,x))$$

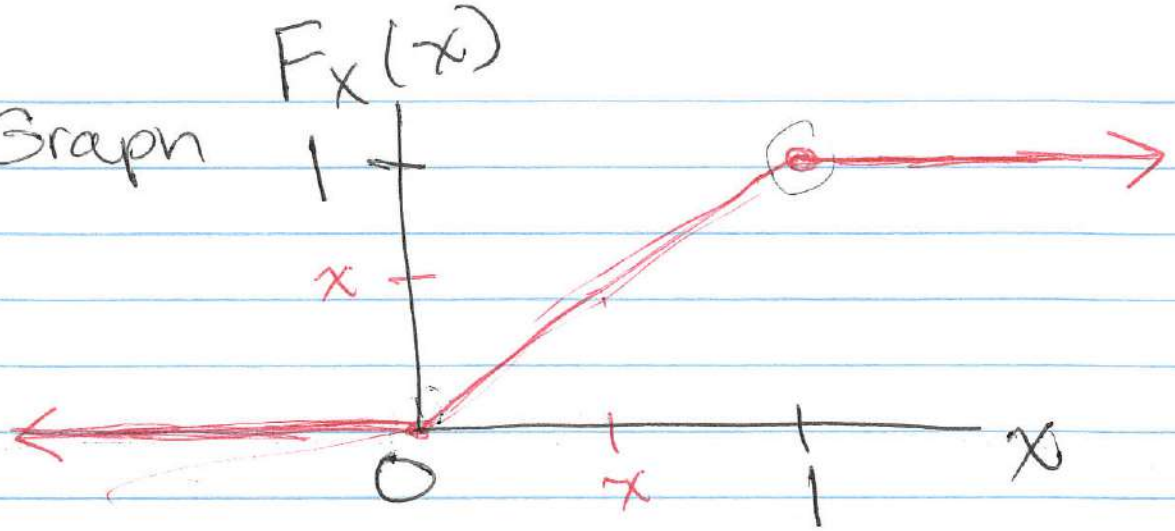
$$= x - 0$$

$$= x$$

CDF of X:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

Graph $F_X(x)$



Properties of a CDF

Let X be a RV with CDF $F_X(x)$ then

① $F(a) \leq F(b)$ if $a \leq b$
(F is nondecreasing)

② $\lim_{x \to -\infty} F_X(x) = 0$

③ $\lim_{x \to +\infty} F_X(x) = 1$

④ $\lim_{x \downarrow x_0} F(x) = F(x_0)$ (Right Continuous)

ex) $\lim_{x \downarrow 1} F(x) = F(1) = \frac{1}{6}$ die roll

$\lim_{x \to 1} F(x) = DNE$ die roll

$\lim_{x \to 1^+} F(x) = \frac{1}{6}$ die roll

(5) $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$

for $a < b$

ex) Let X be the lifetime in yrs of a mechanical part.

Define CDF to be

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & 0 \leq x \end{cases}$$

Find the prob that a part has a lifetime between 1 and 3 yrs.

SOL $\mathbb{P}(1 < X \leq 3) = F_X(3) - F_X(1)$

$= \mathbb{P}(X \leq 3) - \mathbb{P}(X \leq 1)$

$= (1 - e^{-3}) - (1 - e^{-1})$

$= \boxed{e^{-1} - e^{-3}}$

Continuous $= \int_1^3 f(x)\,dx$

$= \mathbb{P}(1 \leq X \leq 3) = \mathbb{P}(1 < X < 3)$

(6) $\mathbb{P}(X = x) = F_X(x) - F_X(x-)$

for all $x \in \mathbb{R}$

$$F_X(x-) = \lim_{z \uparrow x} F_X(z)$$

i.e. $\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x)$

Other Notes:

For continuous RVs:

From pdf to CDF

$$\int_{-\infty}^{x} f(t)\,dt = F(x)$$

↑ lower bound depends on problem.

From CDF to PDF

$$\frac{d}{dx} F(x) = f(x)$$

$f(x) = \frac{d}{dx}\frac{x+3}{9} = \frac{1}{9}$

Ex 11 $F(x) = \begin{cases} 0, & x < 1 \\ \frac{x+3}{9}, & 1 \le x < 3 \\ 1, & 3 \le x \end{cases}$

a) Discrete or Continuous?

$f(x) \propto p^{d-f}$, $f(x) \cdot \frac{1}{9}$, $\int_{3/2}^{5/2} f(x)\,dx$

c) $P\left(\frac{3}{2} < X \le \frac{5}{2}\right)$

$= F\left(\frac{5}{2}\right) - F\left(\frac{3}{2}\right)$

$= \frac{2.5+3}{9} - \frac{1.5+3}{9} = \frac{1}{9}$

d) $P(X=2) = F(2) - F(2-)$

$= \frac{2+3}{9} - \lim_{x \uparrow 2} \frac{x+3}{9}$

$= \frac{5}{9} - \frac{5}{9} = 0$

e) $P(X=3) = F(3) - F(3-)$

$= 1 - \frac{3+3}{9} = \frac{1}{3}$