Stat 401 9/18/17

HW due 9/25 : Sec 1.6 # 2,7,8 Sec 1.7 # 6a, 14, 24
Sec 1.8 # 2, 6

Sec 1.6 - Discrete RVs

def) Discrete RV
- A RV is discrete if it space is either finite or countable.

def) Probability Mass Function (PMF)

Let X be a discrete RV with space $\mathcal{D}$. The pmf of X is given by

$$p_X(x) = \mathbb{P}[X = x] \quad \text{for} \quad x \in \mathcal{D}$$

Properties of a PMF

1. $0 \leq p_X(x) \leq 1$
2. $\sum_{x \in \mathcal{D}} p_X(x) = 1$ ← script D
3. $\mathbb{P}_X(D) = \sum_{x \in D} p_X(x)$ ← capital D

ex) (Geometric Distribution)

Consider a sequence of independent flips of a coin.
Assume $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$

Define a RV X to be

X = # flips needed to get the first head.

$\rightarrow X(H\,H\,T\,T\,H) = 1$

$X(T\,T\,T\,H\,H\,T) = 4$

a) Define space.

$$\mathcal{D} = \{1, 2, 3, 4, \dots\}$$

b) Find $\mathbb{P}[X = 2] = \mathbb{P}_X(2)$

$\underset{\frac{1}{2}}{\underline{T}}$ $\underset{\frac{1}{2}}{\underline{H}}$

$$\mathbb{P}(T\,H) \overset{\text{independent}}{=} \mathbb{P}(T) \cdot \mathbb{P}(H)$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

③

c) $\mathbb{P}[X = n]$ for $n \in D$

$$\underbrace{\underline{T}\ \underline{T}\ \underline{T}\ \underline{T}\ \underline{T}\ \underline{T}}_{\text{all tail.}} \cdots \underset{n}{\underline{H}}$$

n-1 are tail 1-head

$$\mathbb{P}[X=n] = \mathbb{P}_X(n) = \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right)^{1}$$

$$= \left(\frac{1}{2}\right)^{n}, \ n \in D$$

d) Find the CDF

$$F_X(x) = \mathbb{P}[X \le x]$$

$$= \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{1}{2}+\frac{1}{4}, & 2 \le x < 3 \\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$

$\mathbb{P}[X=1] + \mathbb{P}[X=2]$

e) Find prob that 1st head appears on an odd # of flips.

$$\mathbb{P}\left[X \in \{1,3,5,7,\ldots\}\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n}$$

Geometric Series.

$$= \frac{\frac{1}{2}}{1-\frac{1}{4}} = \boxed{\frac{2}{3}}$$

$$\sum_{n=0}^{\infty} a \cdot r^{n} = \frac{a}{1-r}$$

f) Suppose $\mathbb{P}(H) = \frac{1}{3}$, $\mathbb{P}(T) = \frac{2}{3}$.

$$\mathbb{P}[X=n] = \left(\frac{2}{3}\right)^{n-1}\left(\frac{1}{3}\right)$$

$\mathbb{P}[\,X \in \{1, 3, 5, 7, \ldots\}\,]$.

$\underline{\text{sequence}}$

If $X = 1$, H   0 T   1 H

If $X = 3$, T T H   2 T   1 H

If $X = 5$, T T T T H   4 T   1 H

$\uparrow$ even #s.

$$\mathbb{P}[X \in \{1, 3, 5, 7, \ldots\}]$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{2n} \left(\frac{1}{3}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^{n} \left(\frac{1}{3}\right)$$ Geometric Series

$$= \frac{\frac{1}{3}}{1 - \frac{4}{9}} = \boxed{\frac{3}{5}}$$

ex) (Hypergeometric Dist)

100 fuses ← inspected.
5 chosen to inspect.

If all good, then 100 fuses accepted.

Suppose 20 defectives out of 100 fuses

a) Find prob. the 100 fuses are accepted.

$$\mathbb{P}[\text{accept}] = \frac{\binom{80}{5}}{\binom{100}{5}}$$

b) Define a RV X to be

X: # defective fuses among the 5 inspected

defective   not defective

$$D = \{0,1,2,3,4,5\}$$

PMF

$$P_X(x) = \begin{cases} \dfrac{\binom{20}{x}\binom{80}{5-x}}{\binom{100}{5}} & x = 0,1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

Transformations

Suppose $\underset{\substack{\uparrow \\ \text{discrete RV.}}}{Y} = g(X)$ ← function.

Goal: Find the pmf (distribution) of Y.

* Determine the space of Y.

D_X = space of X

D_Y = space of Y

$= \{ g(x) : x \in D_X \}$

* Find PMF of Y.

Case 1: g is a 1-1 function.

$$
\begin{aligned}
p_Y(y) &= \mathbb{P}[Y = y] \\
&= \mathbb{P}[g(X) = y] \\
&= \mathbb{P}[X = g^{-1}(y)] \\
&= p_X(g^{-1}(y))
\end{aligned}
$$

Case2 : Not 1-1

→ no overall rule
→ develop based on context.

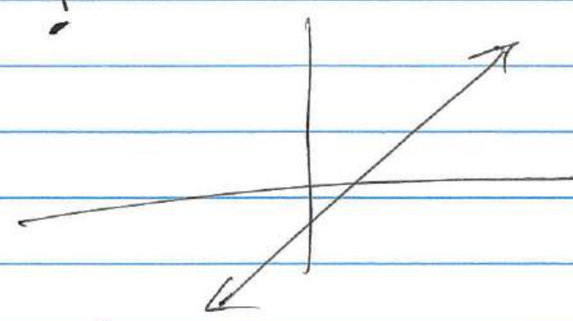
ex) Geomet. Dist with $P(H) = P(T) = 1/2$.

$$p_{X}(x) = \begin{cases} \left(\frac{1}{2}\right)^{x}, & x = 1, 2, 3, \ldots \\ 0, & \text{o.w.} \end{cases}$$

Let Y = # flips before 1st head.

a) Define $g(X) = Y$

$Y = X - 1$ ← 1-1 function



b) Find $D_Y = \{0, 1, 2, \ldots\}$

c) PMF

$$y = x - 1$$

$$y + 1 = x = g^{-1}(y)$$

$$P_Y(y) = p_X(\underline{g^{-1}(y)})$$

$$= p_X(y+1)$$

$$= \begin{cases} \left(\frac{1}{2}\right)^{y+1}, & y = 0, 1, 2, \ldots \\ 0, & \text{o.w.} \end{cases}$$