Stat 401 - 9/20/17

Sec 1.6 - Discrete RV

<u>ex)</u> Geometric Dist. $D = \{1, 2, 3, \ldots\}$

Let $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. X = # flips needed to get 1st head.

$$\mathbb{P}[X = n] = \left(\tfrac{1}{2}\right)^n, \quad n \in D$$

odd # of flips

$$\mathbb{P}[X \in \{1, 3, 5, 7, \ldots\}] = \tfrac{2}{3} \quad ☆$$

Take the transformation Y where:

- suppose we win \$1 if the 1st head appears on an even # of flips
- suppose we lose \$1 if the 1st head appears on an odd # of flips

$D_Y = \{-1, 1\}$.

$$Y = g(X) = \begin{cases} -1, & \text{if } x \text{ is odd} \quad x = 1, 3, 5, 7, \ldots \\ 1, & \text{if } x \text{ is even} \quad x = 2, 4, 6, 8, \ldots \end{cases}$$

Is $g(X)$ a 1-1 function? NO

PMF for Y: $p_Y(-1) = \mathbb{P}[Y = -1]$
$= \mathbb{P}[X \in \{1, 3, 5, 7, \ldots\}]$
$= 2/3$

$p_Y(1) = \mathbb{P}[X \in \{2, 4, 6, 8, \ldots\}]$
$= 1 - 2/3 = 1/3$

$$p_Y(y) = \begin{cases} \frac{2}{3} & \text{if } y = -1 \\ \frac{1}{3} & \text{if } y = 1 \\ 0, & \text{o.w.} \end{cases}$$

In Lecture Notes, review Example 5 for another ex of a function not necessarily 1-1.

Sec 1.7 - Continuous RVs

def) Cont. RV.

- A RV is continuous if its CDF, $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$ (no discontinuities).

- we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$$= \mathbb{P}[X \leq x]$$

$$= \mathbb{P}[-\infty < X \leq x]$$

def) Probability Density Function (PDF)

Let X be a continuous RV. The pdf of X is given by $f_X(x)$.

Properties of a PDF

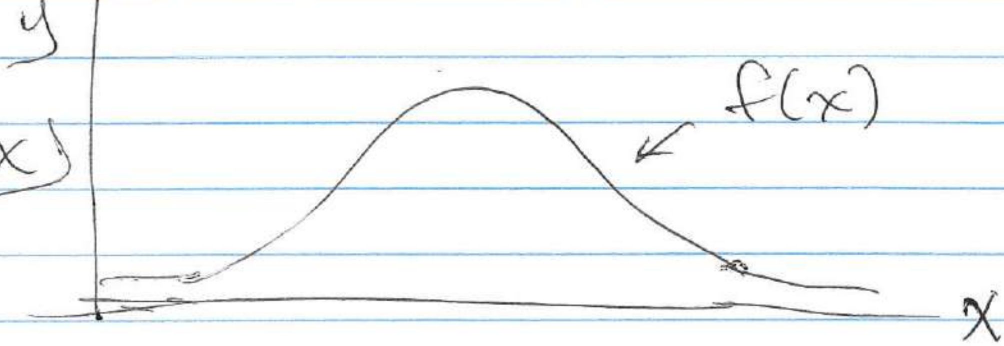
① $f_X(x) = \frac{d}{dx} F_X(x)$ if $f_X(x)$ is continuous

② $f_X(x) \geq 0$ for all $-\infty < x < \infty$

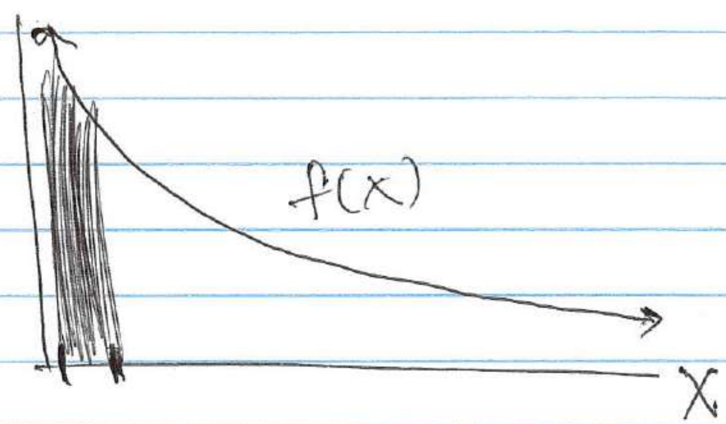
3) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

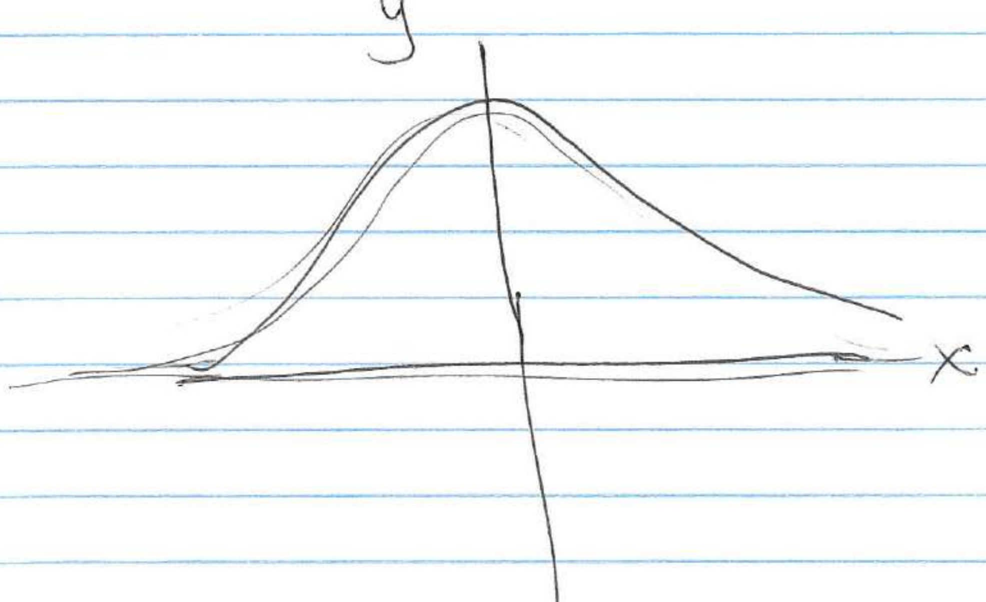
Area under curve $= 1$.

ex)



ex)





4) $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$

If continuous $= \mathbb{P}[a \leq X \leq b]$

$= \mathbb{P}[a < X < b]$

$= \mathbb{P}[a \leq X < b]$

$= \int_a^b f_X(x)\,dx$

def) Support of a continuous RV X

→ consists of all points x such that

$f_X(x) > 0$

Denote by S

Most times $S = D$
But not always.

It is always true that $S \subset D$

ex] Continuous Uniform Dist.

Select a # at random from some interval $[a, b]$

X represents the outcome.

1. Space of X.

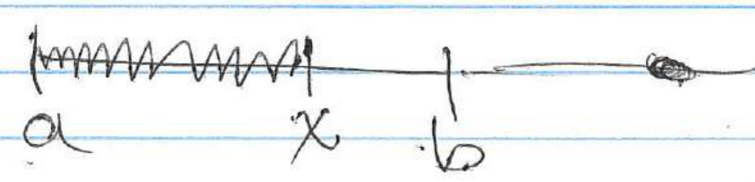
$$D_X = [a, b] = \{x : a \le x \le b\}$$

2. Find CDF.

Case 1: If $x < a$

$$F_X(x) = \mathbb{P}(X \le x) = 0$$
$$= \mathbb{P}(X < a)$$

Case 2: If $a \le x < b$

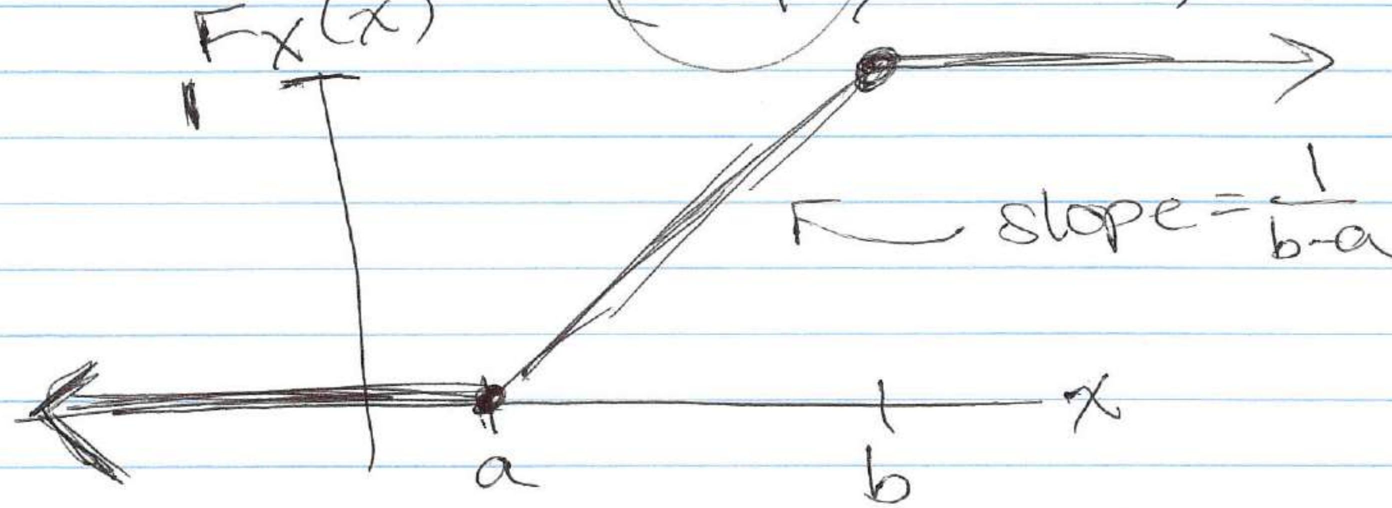


$$\mathbb{P}[X \le x] = \frac{x-a}{b-a}$$

Case 3: If $x \ge b$

$$\mathbb{P}(X \le x) = \mathbb{P}(X \le b+\varepsilon) = 1$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & b \le x \end{cases}$$

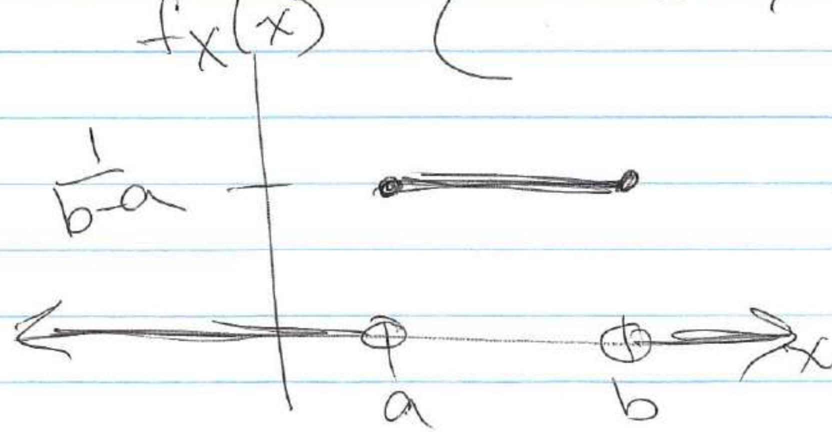


③ Find PDF. $D = S = [a, b]$

$\frac{d}{dx} 0 = 0$ $\frac{d}{dx} \frac{x-a}{b-a} = \frac{1}{b-a}$

$\frac{d}{dx} 1 = 0$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{o.w.} \end{cases}$$



$\int_a^b \frac{1}{b-a} \, dx = 1$

space is where $f(x) \geq 0$
support " " $f(x) > 0$

Transformations

Suppose X is a cont. RV.
Let $Y = g(X)$ be another cont. RV.

Goal: Find the pdf of Y.

* Determine support/space of Y
* Find the CDF of Y
* Use CDF to find PDF.

ex) Suppose $f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{o.w.} \end{cases}$

Let $Y = X^2$

* Look at $-1 < x < 1$
  $0 \leq y < 1$ ← space for Y

* CDF of Y

$$\mathbb{P}[Y \leq y] = \mathbb{P}[X^2 \leq y]$$
$$= \mathbb{P}[-\sqrt{y} \leq X \leq \sqrt{y}]$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx = \sqrt{y}$$

CDF $F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 \le y < 1 \\ 1, & 1 \le y \end{cases}$

* Find PDF.

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 \le y < 1 \\ 0, & \text{o.w.} \end{cases}$$

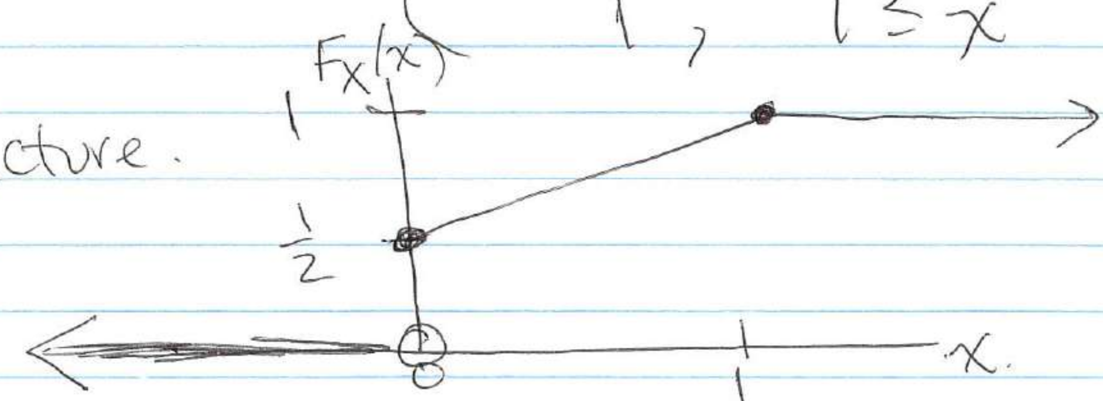
<u>Mixture of RVs</u>

→ combines discrete + continuous.

<u>ex)</u>

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x+1}{2}, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

(1) Picture.



(2) Find $\mathbb{P}[-3 < X \le \frac{1}{2}]$

$$= F(\tfrac{1}{2}) - \underbrace{F(-3)}_{0}$$

$$= \frac{3}{4}$$

(3) Find $\mathbb{P}[X = 0] = F(0) - F(0-)$

$$= \frac{1}{2} - 0 = \frac{1}{2}.$$

Ex 2 - Exp. Dist

Ex 4 - Transformation.