Stat 401 9/22/17

Exam 1 on Friday Oct 6.
Covers Ch 1.
Study guide available at 11 am today
→ Extra problems to help give you practice over the material.
→ Also study lecture notes, homework.
Review Day on Wed Oct 4.

Sec 1.8 - Expectation of a RV

def) Expected Value of X

If X is a discrete RV with pmf $p_X(x)$, then expectation of X is

$$\mu_X = \mathbb{E}[X] = \sum_{x \in D_X} x \cdot p_X(x)$$

Expectation in general

Other names:
- mean value of X
- mean value of distribution
- First moment
- arithmetic mean of values of X

Continuous case with pdf $f_X(x)$

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

ex] Suppose X has pmf.

x^2	1	4	9	16
x	1	2	3	4
$p_X(x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

$$\mathbb{E}[X] = \sum_{x=1}^{4} x \cdot p_X(x)$$

$$= 1\left(\frac{4}{10}\right) + 2\left(\frac{1}{10}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{2}{10}\right)$$

$$= \frac{23}{10} = 2.3$$

1 1 1 1 2 3 3 3 4 4

$\Sigma = 23$ $n = 10$

Avg $= 2.3$

ex] Let X have pdf.

$$f_X(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X] = \int_0^1 x \cdot 4x^3\,dx = \int_0^1 4x^4\,dx$$

$$= \boxed{\frac{4}{5}}$$

Functions of X

def) Expected Value of $g(X)$ ← function

Let $g(X)$ be a function of RV X, then

$$\mathbb{E}[g(X)] = \begin{cases} \sum_{x \in D_X} g(x)\, p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

ex) $\mathbb{E}[X^2] = \sum_{x} x^2 p_X(x)$ or $\int_{-\infty}^{\infty} x^2 f(x)\,dx$

$g(X) = X^2$

$$\mathbb{E}[aX^2+bX] = \sum_x (ax^2+bx)\,p_X(x)$$

$$= \int_{-\infty}^{\infty} (ax^2+bx)\,f_X(x)\,dx$$

$$\mathbb{E}[\underset{\substack{\uparrow \\ \text{constant}}}{k}] = k$$

Proof) (discrete)

$$\mathbb{E}[k] = \sum_x k \cdot p_X(x)$$

$$= k \underbrace{\boxed{\sum_x p_X(x)}}_{=1}$$

$$= k$$

ex) Let X have pdf

$$f_X(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$$

Find $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 2(1-x)\,dx = \frac{1}{6}$$

ex) Bowl - has 5 chips.
3 are marked \$1 each
2 are marked \$4 each.

Player draws 2 chips at random and without replacement.

The player is paid an amount equal to the sum of the values of the 2 chips that he draws.

$\boxed{Q}$ Find the expected value of the amount won by the player.

$\boxed{SOL}$ $X = \#$ of \$4 chips player draws

$$D_X = \{0, 1, 2\}$$

$$\text{Amount won} = \underbrace{X}_{\#\$4\text{ chips}} \cdot \underbrace{4}_{\text{amt won}} + \underbrace{(2-X)}_{\#\$1\text{ chips}} \cdot \underbrace{1}_{\text{amt won}}$$

$$= 3X + 2$$

Find $E[3X+2]$

ways to get $4 chips

$$p_X(0) = \mathbb{P}[X=0] = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$$

ways to get 2 chips from 5.

$$p_X(1) = \mathbb{P}[X=1] = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10}$$

$$p_X(2) = \mathbb{P}[X=2] = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}$$

$$\mathbb{E}[\underbrace{3X+2}_{\$4+\$1}] = \sum_{x=0}^{2} (3x+2)\,p_X(x)$$

$$= (3\cdot 0+2)\left(\frac{3}{10}\right) + (3\cdot 1+2)\left(\frac{6}{10}\right) + (3\cdot 2+2)\left(\frac{1}{10}\right) = \boxed{\frac{23}{5}}$$

Thm) Let $g_1(X)$ and $g_2(X)$ be functions of RV X. Let k_1 and k_2 be constants.

Then

$$\mathbb{E}\left[k_1 g_1(X) + k_2 g_2(X)\right]$$

$$= k_1 \mathbb{E}[g_1(X)] + k_2 \mathbb{E}[g_2(X)]$$

ex) $\mathbb{E}[6X + 3X^2] = 6\mathbb{E}[X] + 3\mathbb{E}[X^2]$

$$\mathbb{E}[-4X^3 - 2X]$$

$$= -4\mathbb{E}[X^3] - 2\mathbb{E}[X]$$

ex) RV X with prob. dist

x	0	1	3
$P_X(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

discrete.

a) $\mathbb{P}[X = 2] = 0$

b) $\mathbb{P}[X < 2] = \frac{5}{6}$

c) $\mathbb{E}[X] = \sum_{x=0}^{3} x P_X(x)$

$$= 0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{6}\right) = 1$$

d) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x)$

$= 0^2\left(\frac{1}{3}\right) + 1^2\left(\frac{1}{2}\right) + 3^2\left(\frac{1}{6}\right)$

$= 2$

e) Find $\mathbb{E}[Y]$ where $Y = (X-1)^2$.

$\mathbb{E}[Y] = \mathbb{E}\left[(X-1)^2\right]$

Opt 1 $= \sum_x (x-1)^2 p_X(x)$

Opt 2 $= \mathbb{E}\left[X^2 - 2X + 1\right]$

$= \underline{\mathbb{E}[X^2]} - 2\underline{\mathbb{E}[X]} + 1$

$= 2 - 2(1) + 1$

$= 1$

Note: Find μ_X ← Find $\mathbb{E}[X]$

Find μ_{X^2} ← Find $\mathbb{E}[X^2]$