Stat 401 9/25/17

HW due 10/2/17 : Sec 1.9: 1b, 2, 7
Sec 1.9: 16 ← use pmf to find mean and variance
Sec 1.10 : 2, 3
Sec 2.1 : 9, 10

Sec 1.9 - Some Special Expectations.

Recall) The Mean of a RV X is

$$\mu_X = \mathbb{E}[X]$$

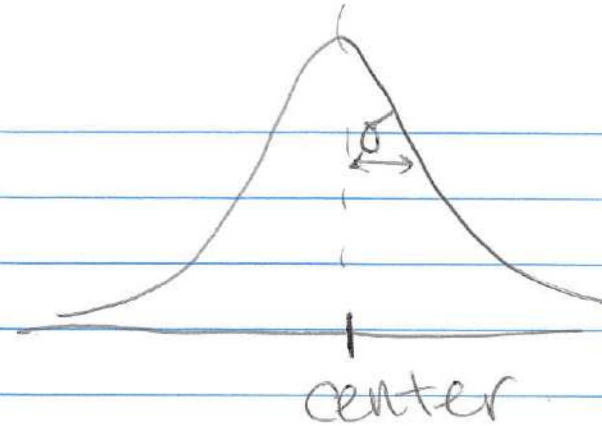
→ center of distribution.

def) Variance of a RV X

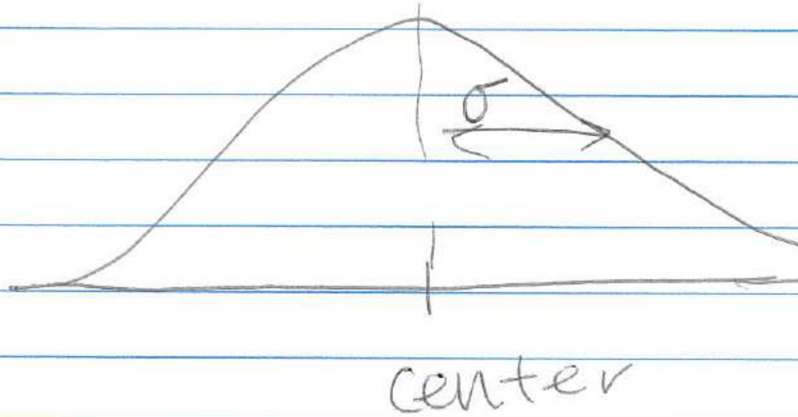
$$\sigma^2 = \sigma_X^2 = Var(X) = \mathbb{E}[(X-\mu_X)^2] \geq 0$$

Interpretation:
- small variance means data/pts are more centrated around mean (center).
- large variance means, data more dispersed/spread out.

ex)



σ^2 smaller.



σ^2 larger

How to compute:

$$\sigma^2 = \mathbb{E}\left[(X-\mu)^2\right] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

$$= \mathbb{E}\left[X^2 - 2\mu X + \mu^2\right]$$

Note: μ is a #

$$= \mathbb{E}[X^2] + \mathbb{E}[-2\mu X] + \mathbb{E}[\mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

def: Standard Deviation of X.

- the measure of the dispersion of the points of the space relative to mean value

$$\sigma_X = SD(X) = \sqrt{Var(X)} \geq 0$$

ex: Let X have pdf

$$f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

Find μ, σ^2, σ

$\boxed{\text{SOL}}$ $\mu = \mathbb{E}[X] = \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\, dx$

$$= \frac{1}{3}$$

$$\sigma^2 = Var[X] = \mathbb{E}\left[\left(X - \overset{\mu}{\frac{1}{3}}\right)^2\right]$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Aside: $\mathbb{E}[X^2] = \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1)\, dx = \frac{1}{3}$

$$\sigma^2 = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \boxed{\frac{2}{9}}$$

$$\sigma = SD(X) = \sqrt{\frac{2}{9}} = \boxed{\frac{\sqrt{2}}{3}}$$

Moment Generating Function (MGF)

def | MGF

The MGF of X is defined to be the function

$$\underbrace{M}_{\text{MGF}}\,{}_{\underbrace{X}_{\text{RV}}}\underbrace{(t)}_{\text{function in } t} = \underbrace{\mathbb{E}\left[e^{\overset{\text{constant.}}{t}X}\right]}_{\text{Function in } X}$$

ex | [Exponential Dist with parameter $\theta = 1$]
Let X have pdf

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

MGF:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \cdot e^{-x} \, dx$$

$$= \int_0^\infty e^{(t-1)x} \, dx = \int_0^\infty e^{-(1-t)x} \, dx$$

$$= -\frac{1}{1-t} e^{-(1-t)x} \Big|_0^\infty$$

Caution → want a finite expectation.

we need to restrict t and have $t<1$ (or else expectation is not finite).

$$= -\frac{1}{1-t}\left(0 - e^0\right) = -\frac{1}{1-t}(-1)$$

$$= \frac{1}{1-t}$$

$$M_X(t) = \frac{1}{1-t} \text{ for } t<1$$

ex) [Exponential Dist for General parameter θ]

Let X have pdf

$$f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

for $\theta > 0$ (a scaling factor).

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

$$= \frac{1}{\theta} \int_0^\infty e^{-(\frac{1}{\theta} - t)x} dx$$

$$= \frac{1}{\theta} \cdot \frac{-1}{\frac{1}{\theta} - t} e^{-(\frac{1}{\theta} - t)x} \Big|_0^\infty$$

Constraint on t: $t < \frac{1}{\theta}$

$$= \frac{-1}{1-\theta t}(0 - e^0) = \frac{1}{1-\theta t}$$

for $\theta > 0$ and $t < \frac{1}{\theta}$

ex) $M_X(t) = \dfrac{1}{1-2t}$ for $t < \frac{1}{2}$

Identify pmf / pdf.

$\boxed{\text{SOL}}$ $\theta = 2$

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

def) Moments of a RV X.

$$\mathbb{E}[X^k] = k^{th} \text{ moment of } X$$
$$= M_X^{(k)}(0)$$

RV — k^{th} derivative — evaluate at $t=0$

0^{th} Moment: $M_X^{(0)}(0) = \mathbb{E}[e^{0 \cdot X}]$

$= \mathbb{E}[1] = 1$

1st Moment: $M_X^{(1)}(0) = \mathbb{E}[X]$.

Proof)

$$M'(t) = \frac{d}{dt}\mathbb{E}\left[e^{tX}\right]$$

$$= \mathbb{E}\left[\frac{d}{dt} e^{tX}\right]$$

$$= \mathbb{E}\left[X e^{tX}\right]$$

$$M'(0) = \mathbb{E}\left[X e^{0 \cdot X}\right] = \mathbb{E}[X]$$

2nd Moment $M_X^{(2)}(0) = \mathbb{E}[X^2]$

Variance: $\sigma^2 = Var(X)$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= M''(0) - (M'(0))^2$$

ex) $f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$

$M_X(t) = \frac{1}{1-t} = (1-t)^{-1}$ for $t<1$

$M_X'(t) = -1(1-t)^{-2}(-1) = (1-t)^{-2}$

$M_X'(0) = (1-0)^{-2} = 1 = \mathbb{E}[X]$

$M_X''(t) = -2(1-t)^{-3}(-1) = \frac{2}{(1-t)^3}$

$M_X''(0) = 2 = \mathbb{E}[X^2]$

$\sigma^2 = \text{Var}(X) = 2 - 1^2 = 1$